Journal of Statistical Physics, Vol. 47, Nos. 1/2, 1987

Theory of Quantum Dynamics in Fermionic Environment: An Influence Functional Approach

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Received October 7, 1986

Quantum dynamics of a particle coupled to a fermionic environment is considered, with particular emphasis on the formulation of macroscopic quantum phenomena. The framework is based on a path integral formalism for the real-time density matrix. After integrating out of the fermion variables of the environment, we embed the whole environmental effects on the particle into the so-called influence functional in analogy to Feynman and Vernon's initial work. We then show that to the second order of the coupling constant, the exponent of the influence functional is in exact agreement with that due to a linear dissipative environment (boson bath). Having obtained this, we turn to a specific model in which the influence functional can be exactly evaluated in a long-time limit (long compared to the inverse of the cutoff frequency of the environmental spectrum). In this circumstance, we mainly address our attention to the quantum mechanical representation of the system-plus-environment from the known classical properties of the particle. It is shown that, in particular, the equivalence between the fermion bath and the boson bath is generally correct for a single-channel coupling provided we make a simple mapping between the nonlinear interaction functions of the baths. Finally, generalizations of the model to more complicated situations are discussed and significant applications and connections to certain practically interesting problems are mentioned.

KEY WORDS: Path integral; influence functional; fermion bath; real-time density matrix; Grassmann algebra; quantum Langevin equation.

1. INTRODUCTION

It has been an interesting question whether a macroscopic quantity can exhibit quantum properties that can be measured from experiments. By *macroscopic*, in fact, different people have different interpretations. What is

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0022-4715/87/0400-0017$05.00/0

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more important is rather that a macroscopic quantity usually consists of a large degree of freedom at certain microscopic level(s). Thus, one important point[2] to look into is whether this large amount of freedom can destroy the effects of the superposition principle, which is assumed to be a universal rule for our world. In a narrow sense, the problem can be cast into the question how the environment—the freedom at a certain microscopic level(s)—affects the quantum dynamics or the quantum properties of their macroscopically collective mode. For instance, one can ask whether there are possibly macroscopic quantum tunneling (MQT) and macroscopic quantum coherence (MQC). Therefore, the initial question is essentially a problem of the dynamics of many-particle quantum mechanics. Also, because one is interested in the dynamics of the macroscopically collective mode rather than certain thermal or average properties of the macroscopic system, the major source of the difficulties for the problem lies in the fact that there are too many uncertainties for the environment, possessing a large degree of microscopic freedom. Besides, the information concerning the interactions between the quantity of interest and its environment is quite limited in most practical problems.

It is well known that coupling between a system and its environment usually results in an irreversible energy dissipation. In classical theory, this dissipation is commonly represented by introducing certain dissipation terms into the equation(s) of motion, most frequently a dissipation term linear in velocity. However, if this is the only information one knows for the classical correspondence, there is no unique scheme to formulate the quantum description for the whole system (system-plus-environment). Thus, a fundamental problem is how, in a way as natural as possible, to truncate the basically unknown environment and its interactions with the bare system into a proper form upon which quantum mechanical calculations can be performed.

Fortunately, there are still some ways to deal with the above situation. It was well conjectured that if the motion of the system only weakly perturbs the environment, in other words, the coupling between the system and each of the microscopic freedoms of the environment is so small that a second-order perturbation is adequate to account for its effects, then the environment can be effectively cast into a set of harmonic oscillators while one considers its effects on the dynamics of the system.[1,2] Note that this is by no means to say that the whole effect due to the environment is necessarily small. This offers a natural, phenomenological-like scheme to

[2] The other point, which cannot be answered within the framework of quantum theory, is whether the quantum mechanical description of a macroscopic quantity is correct; investigations concerning this point can be found elsewhere. However, this is beyond the scope of this work.

solve this problem of many-particle quantum mechanics, as is described below.[2] One first models the complete system (system-plus-environment) by truncating the environment into a harmonic oscillator bath coupled to the system. Then one determines the coupling terms via a correspondence to the known classical properties of the system such as equation(s) of motion, under the limit of $\hbar \to 0$. After this, various quantum mechanical calculations are readily carried out.

Even a simple quantum system can have profound changes when it is coupled to a dissipative environment. Historically, Feynman and Vernon considered a general coordinate-like quantum system coupled to a linear dissipative environment represented by a set of harmonic oscillators.[1] Using their path integral approach to quantum mechanics, they were able to put the whole environmental effect on the system into the famous Feynman influence functional. Later, many studies in the topics of quantum Brownian motion[3] and quantum Langevin equation[4] were made based on this approach. Recently, this functional integral approach has been extended into imaginary time to calculate the rate of MQT out of a metastable potential.[3] Also, the Feynman method has been used for a direct calculation of the MQC properties of a dissipative two-state system, with fruitful results (cf. Ref. 6, where many other references on this subject can be found).

However, from the statistical physics point of view, there are profound differences between the Fermi statistics and Bose statistics. Although in the above-described weak coupling limit any environment can be approximated by a harmonic oscillator bath of bosonic nature, one may still ask how it will be if the environment considered is actually fermionic and the coupling is not necessarily weak. Part of the answer already exists in the literature. Hamann[7] and later Yu and Anderson,[8] in an imaginary-time functional approach, considered the partition function of an impurity-like particle surrounded by an electron gas. In their work, a saturation of the effective coupling between the particle and the electrons was found, indicating the significance of the environmental nonlinearity. As regards real-time dynamics, Chang and Chakravarty[9] did an explicit calculation in a two-state system coupled to either a Bose or a Fermi gas. Their results showed that the coupling to a fermion bath is equivalent to the coupling to a boson bath for the particular system provided one makes a suitable correspondence for the coupling constants. Recent work by

[3] For these, a quite complete calculation at zero temperature was given in Ref. 2 and a microscopic confirmation on the phenomenological model was given in Ref. 5. Finite-temperature extensions were also worked out by several authors.[22] In addition, some numerical evaluations are available.[23] Finally, a recent review by Hanggi[24] covers the generally topics of interest in this area.

Hedegård and Caldeira[10] gave, among other things, a generalization of Chang and Chakravarty's result to a multistate, though still discretized, system. They also gave a detailed introduction to the current situation and the significance of the dissipative dynamics in a fermionic environment.

Nevertheless, of great importance in practice, a more generalized question should be proposed concerning a general system coupled to a fermionic environment: Is there any systematic or unified way to predict possible quantum properties simply from the classical correspondences, as people have done for the boson bath coupling? Moreover, it is of particular interest whether the general conjecture mentioned above does stand for the fermion bath coupling, more precisely, whether the fermion bath can be effectively cast into the boson bath via some suitable correspondences. This is, in fact, precisely the problem we are looking into in this work.

We basically employ an extended path integral approach (thus forming a continuous state system) to the problem by virtue of Grassmann algebra developed particularly for a fermion field theory.[11] On arriving at this point, we first adopt a modified Keldysh technique[12] to describe the real-time density matrix of the particle in a complex-time contour known as the Baym–Kadanoff contour.[13] Then we demonstrate explicitly the transformation of the expression into a path integral representation. This makes our approach fall into that of Feynman and Vernon, where a formal path integral defined in Grassmann algebra represents the effects of the fermionic environment.

This paper is arranged as follows. In Section 2, we offer a formal path integral representation for the real-time density matrix, which serves as a fundamental formalism for the rest of our work. In Section 3, we evaluate the function $J(x, y, t; x', y', 0)$ [see below, Eq. (6)] explicitly to the second order of the coupling constant. The result shows an equivalence between the boson and fermion baths. In particular, when coupling coefficients are taken to be an overall constant, an unambiguous Ohmic spectrum appears. In Section 4, this case is further explored. An exact solution in the long-time limit (long compared to the inverse of the cutoff frequency of the environment) is found by virtue of a method first introduced by Nozières and De Dominicis[14] in the discussion of the X-ray edge problem in metals (a slight extension to a multichannel coupling was made there, too). In Section 5, the quantum Langevin equation is derived from that model with full nonlinearities involved. We thus obtain a way to predict the quantum properties from the corresponding classical equation of motion. It turns out that the results for a single channel coupling are equivalent to those of a boson bath coupling provided we make a suitable mapping between the nonlinear interaction functions in the two baths. Finally, in Section 6, we discuss some possible generalizations of the model to more complicated

situations, and mention the connections to other interesting circumstances. Appendix A is devoted to the detailed derivation of the path integral representation in Section 2, while Appendix B is concerned with the full derivation and justification of the solution used in Section 4, and Appendix C gives the effective Euclidean action for the fermionic environment for the convenience of MQT calculations.

2. PATH INTEGRAL FORMALISM FOR THE REAL-TIME DENSITY MATRIX

A large variety of quantum systems are surrounded by fermionic environments. Among these, some of the important examples are mentioned in Ref. 9 and 10. Since we are dealing with a fermionic environment, it would be more convenient and physically transparent to work directly on the fundamental fermionic excitations rather than their bosonic combinations. To maintain as many varieties of applications as possible through this work, we begin with the following very generalized Hamiltonian[4]—the system considered is characterized by a particle with continuous coordinate, while the environment is presented in the form of creation and annihilation operators:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) + \sum_{i,j=1}^{\Omega} C_{ij}(\hat{x})\, \hat{b}_i^\dagger \hat{b}_j + \sum_{i=1}^{\Omega} \hat{b}_i^\dagger \varepsilon_i \hat{b}_i + \hat{H}_{\text{int}}(\{\hat{b}_i^\dagger, \hat{b}_i\}) \tag{1}$$

where the notations are standard: $\hat{x}$ ($\hat{p}$) is the coordinate (momentum) operatum) operator of the particle with a mass m moving in an arbitrary potential $V(x)$; $\{\hat{b}_i^\dagger, \hat{b}_i\}$ is the set of creation and annihilation operators for the fermionic heat bath with energy level $\{\varepsilon_i\}$ and an interaction Hamiltonian $\hat{H}_{\text{int}}(\{\hat{b}_i^\dagger, \hat{b}_i\})$ between the fundamental fermions. To study the real-time properties of this kind of general quantum system, the most relevant quantity to look into is always the real-time density operator, given via the definition

$$\hat{\rho}(t) \equiv e^{-i\hat{H}t} \hat{\rho}(0)\, e^{i\hat{H}t} \tag{2}$$

or

$$\hat{\rho}(x, y, t) \equiv \langle x |\, e^{-i\hat{H}t} \hat{\rho}(0)\, e^{i\hat{H}t} \,| y \rangle \tag{2'}$$

where the left side is still an operator with respect to the environment. One has to assign a initial density matrix $\hat{\rho}(0)$ to the complete system (system-

[4] One should notice, however, that we have excluded an important class, namely, systems interacting with their environments via momentum-coupling mechanisms are not considered here. See remarks in Section 6.

plus-environment). Although arbitrary choices can be made, we shall take the most common and most convenient one—the factorized initial density matrix[5] as follows:

$$\hat{\rho}(0) = [\exp(-\beta \hat{H}_e)]\, \hat{\tilde{\rho}}(0) \tag{3a}$$

with

$$\langle x' |\, \hat{\tilde{\rho}}(0)\, | y' \rangle = \tilde{\rho}(x', y', 0) \tag{3b}$$

$$\hat{H}_e = \sum_{i=1}^{\Omega} \hat{b}_i^{\dagger} \varepsilon_i \hat{b}_i + \hat{H}_{\text{int}}(\{\hat{b}_i^{\dagger}, \hat{b}_i\}) \tag{3c}$$

Notice that the right-hand side of Eq. (3b) is no longer an operator, and $\hat{H}_e$ indicates the environmental Hamiltonian as defined in Eq. (1). Upon this assumption, the $\hat{\rho}(x, y, t)$ can be written

$$\hat{\rho}(x, y, t) = \int dx'\, dy'\, \langle x |\, e^{-i\hat{H}t}\, | x' \rangle\, e^{-\beta \hat{H}_e}\, \langle y' |\, e^{+i\hat{H}t}\, | y \rangle\, \tilde{\rho}(x', y', 0) \tag{4}$$

However, because the behavior of the environment is beyond our scope, we are only interested in the reduced density matrix for the particle alone, given by tracing out of the environmental variables,

$$\begin{aligned} \rho(x, y, t) &= \text{Tr}_e\, \hat{\rho}(x, y, t) \\ &= \int dx'\, dy'\, \tilde{\rho}(x', y', 0)\, J(x, y, t; x', y', 0) \end{aligned} \tag{5}$$

where

$$\begin{aligned} J(x, y, t; x', y', 0) &= \text{Tr}_e[\langle x |\, e^{-i\hat{H}t}\, | x' \rangle\, e^{-\beta \hat{H}_e}\, \langle y' |\, e^{+i\hat{H}t}\, | y \rangle] \\ &= \text{Tr}_e[e^{-\beta \hat{H}_e}\, \langle y' |\, e^{+i\hat{H}t}\, | y \rangle \langle x |\, e^{-i\hat{H}t}\, x' \rangle] \end{aligned} \tag{6}$$

It is now a matter of expressing the right-hand side of Eq. (6) in terms of path integral formalism. This is done in Appendix A of this work. We then arrive at the formula

$$\begin{aligned} &J(x, y, t; x', y', 0) \\ &\quad \equiv \int \mathscr{D}z(\tau) \prod_{i=1}^{\Omega} \mathscr{D}b_i^{\dagger}(\tau)\, \mathscr{D}b_i(\tau) \exp\left(i \int_{\gamma} d\tau \left[\frac{1}{2} m\dot{z}^2 - V(z) \right.\right. \\ &\qquad \left.\left. + \sum_{i=1}^{\Omega} b_i^{\dagger}(i\partial_{\tau} - \varepsilon_i)\, b_i - \sum_{i,j=1}^{\Omega} C_{ij}(z)\, b_i^{\dagger} b_j - H_{\text{int}}(\{b_i^{\dagger}, b_i\}) \right] \right) \end{aligned} \tag{7}$$

[5] It should be pointed out, however, that there are no special difficulties in making other choices in our path integral formalism.

where the integral contour γ is the so-called Baym–Kadanoff contour[13] as shown in Fig. 1. The following boundary conditions are imposed on the path integral in Eq. (7):

$$z(0+i\varepsilon)=x', \qquad z(0-i\varepsilon)=y', \qquad b_i(0+i\varepsilon)=-b_i(-i\beta)$$
$$z(t+i\varepsilon)=x, \qquad z(t-i\varepsilon)=y, \qquad b_i^\dagger(0+i\varepsilon)=-b_i^\dagger(-i\beta)$$
$$c_{ij}(z(t))\equiv 0 \qquad \text{for} \quad \tau\in(0-i\varepsilon,\ -i\beta)$$

It ought to be kept in mind that the path integral over the fermion variables $\{b_i^\dagger, b_i\}$ is defined in *Grassmann algebra*, which differs from the ordinary one by imposing an anticommutation relation between any two Grassmann numbers. The integral is, in fact, quite formal and should be thought of more like a definition.

Up to this stage, we have not yet specified any detailed structures for the environment. Thus, one can easily realize that the formula (7) is itself of broad applicability to very general situations because it offers a unified way to treat the problem in both the classical and quantum limits. In the classical limit, one can use standard quasiclassical approximations, such as the stationary phase approximation (see below). On the other hand, in the quantum limit the diagram technique of quantum pertubation theory is readily applicable.[6] In addition, one important feature concerned with the

[6] For this, notice that in Appendix A, instead of transforming Eq. (A7) into a path integral representation, one can expand the interaction exponents in Eq. (A3) into powers of the functional derivatives with respect to the external sources of the fermion bath, while transforming the coordinate part into the well-known path integral formula. In this way, one recovers the quantum perturbation theory.

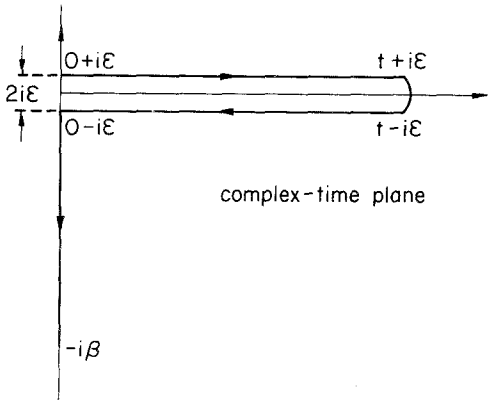

Fig. 1. A schematic representation of the contour γ (Baym–Kadanoff contour). Note that the quantity ε is infinitesimal.

advantages of the path integral technique lies in the fact that in it, unlike the ordinary perturbation theory, it is quite easy to formulate new collective modes. This allows us to treat some interacting fields fairly conventionally.[11] Suppose that we are dealing with a superconducting electron gas; then a pair field[5,11] can be introduced to eliminate the interactions between electrons. This results in a new representation of the environment in which the coupling between the system and the environment becomes a sum of quasiparticle scattering and pair creating and destroying operators. In another important situation, where the environment consists of a normal interacting Fermi liquid, the appropriate new modes would be the quasiparticle excitations and the coupling Hamiltonian would become quasiparticle scattering-like interactions. This point was discussed extensively in Ref. 11, where the reader can find some illustrative examples. As a result, even though our original environment is very complicated, it is always possible to transform it into a simple one in terms of its collective-mode excitation structures. For these reasons, a study of some simple but practically interesting cases will never be meaningless.[7]

In what follows, we shall concentrate on the case in which the particle is coupled via a *separable* scattering-like interaction to a noninteracting Fermi gas; more precisely, $\hat{H}_{\text{int}}(\{\hat{b}_i^\dagger, \hat{b}_i\}) = 0$ and $C_{ij}(\hat{x}) = C_{ij}F(\hat{x})$ in the Hamiltonian (1), with $F(\hat{x})$ an arbitrary function of $\hat{x}$ called the *interaction function* of the coupling Hamiltonian [we shall return to the general situation for nonseparable $C_{ij}(x)$ in the final section, where we mention important generalizations of our results]. Then

$$J(x, y, t; x', y', 0) \equiv \int \mathscr{D}z(\tau) \prod_{i=1}^{\Omega} \mathscr{D}b_i^\dagger(\tau)\, \mathscr{D}b_i(\tau) \exp\left\{ i \int_\gamma \left[\frac{1}{2} m\dot{z}^2 - V(z) + \sum_{i=1}^{\Omega} b_i^\dagger (i\partial_\tau - \varepsilon_i)\, b_i - F(z) \sum_{i,j=1}^{\Omega} C_{ij} b_i^\dagger b_j \right] \right\} \tag{8}$$

At this stage, the integral over the fermion variable can be easily carried out via an integral formula on an exponential bilinear form in Grassmann algebra.[11] It turns out that

$$J(x, y, t; x', y', 0) = \int \mathscr{D}z(\tau) \exp\left\{ i \int_\gamma d\tau \left[\frac{1}{2} m\dot{z}^2 - V(z) \right] - \operatorname{tr} \log \mathbf{G} \right\} \tag{9}$$

where $\mathbf{G}$ is the matrix notation of the full Green's function given by

$$G_{ij}(\tau, \tau') = [(i\partial_\tau - \varepsilon_i)\, \delta_{ij} \delta_\gamma(\tau, \tau') - C_{ij} F(z(\tau))\, \delta_\gamma(\tau, \tau')]^{-1} \tag{10}$$

[7] Chang and Chakarvarty[9] give similar arguments in terms of the well-known Landau quasiparticle picture.

namely,

$$\sum_{k=1}^{\Omega} [(i\partial_\tau - \varepsilon_i)\,\delta_{ik} - C_{ik}F(z(\tau))]\,G_{kj}(\tau, \tau') = \delta_{ij}\delta_\gamma(\tau, \tau') \tag{10'}$$

Notice that $\tau, \tau' \in \gamma$ and $\delta_\gamma(\tau, \tau')$ is the *δ-function* defined along the contour γ. If we now employ the additional matrix notations

$$\mathbf{G}^0 \equiv [(i\partial_\tau - \varepsilon_i)\,\delta_{ij}\delta_\gamma(\tau, \tau')]^{-1}, \quad \mathbf{C} \equiv C_{ij}F(z(\tau))\,\delta_\gamma(\tau, \tau'), \quad \mathbf{I} \equiv \delta_{ij}\delta_\gamma(\tau, \tau')$$

then Eq. (10) or (10′) says

$$\mathbf{G} = (\mathbf{I} - \mathbf{G}^0\mathbf{C})^{-1}\,\mathbf{G}^0 \tag{11}$$

or

$$\mathbf{G} = \mathbf{G}^0 + \mathbf{G}^0\mathbf{C}\mathbf{G} \tag{11'}$$

These are in the form of Dyson's equation. We can use a common trick to treat the trace conveniently. Let

$$\mathbf{G}_g = \mathbf{G}^0 + g\mathbf{G}^0\mathbf{C}\mathbf{G}_g \tag{12}$$

Then

$$\operatorname{tr}\log\mathbf{G} \equiv \operatorname{tr}\log\mathbf{G}^0 + \int_0^1 dg\,\operatorname{tr}\mathbf{G}_g\mathbf{C} \tag{13}$$

The first term in Eq. (13) is independent of $\mathbf{C}$ and contributes only a trivial factor to the normalization of the path integral; cf. Eqs. (A7), (A9), and (A11) for the proper renormalization. Equation (9), together with Eqs. (12) and (13), forms the framework for the rest of this paper. We now turn to detailed analyses of these formulas.

3. SECOND-ORDER APPROXIMATION ON MAPPING TO BOSONIC BATH

As a first approximation to Eq. (12), we keep one more term than just $\mathbf{G}^0$. We then have

$$\mathbf{G}_g = \mathbf{G}^0 + g\mathbf{G}^0\mathbf{C}\mathbf{G}^0 + O(\mathbf{C}^2) \tag{14}$$

$$\operatorname{tr}\log\mathbf{G} = \operatorname{tr}\left\{\mathbf{G}^0\mathbf{C} + \frac{1}{2}\mathbf{G}^0\mathbf{C}\mathbf{G}^0\mathbf{C} + O(\mathbf{C}^3)\right\} \tag{15}$$

In Appendix A, we give a direct evaluation of $\mathbf{G}^0$, which says [Eq. (A8)]

$$G^0_{ij}(\tau, \tau') = -i\delta_{ij}[\theta_\gamma(\tau, \tau') - f(\varepsilon_i)]\exp[-i\varepsilon_i(\tau - \tau')]$$

where $\theta_\gamma(\tau, \tau')$ represents the step function along the contour γ; $f(\varepsilon)$ is the Fermi distribution function. In dealing with the trace, we treat the equal-time limit as $\tau' \to \tau^+$, as usually encountered in the literature (this is most easily understood by visualizing that in the original Hamiltonian the products of $\hat{b}^\dagger$ and $\hat{b}$ always appear in combinations of $\hat{b}^\dagger \hat{b}$). Thus,

$$\begin{aligned}\operatorname{tr} \log \mathbf{G} = \oint d\tau\, z(\tau) \sum_{i=1}^{\Omega} if(\varepsilon)\, C_{ij} - \frac{1}{2} \oint d\tau\, d\tau' \Big\{ F(z(\tau))\, F(z(\tau')) \\ \times \sum_{i,j=1}^{\Omega} \{\exp[-i(\varepsilon_i - \varepsilon_j)(\tau - \tau')]\} [\theta_\gamma(\tau, \tau') - f(\varepsilon_i)] \\ \times C_{ij} C_{ji} [\theta_\gamma(\tau' - \tau) - f(\varepsilon_j)] \Big\} \end{aligned} \tag{16}$$

where the integral over the loop $\oint$ means integrating from $0 + i\varepsilon$ to $t + i\varepsilon$, then from $t - i\varepsilon$ back to $0 - i\varepsilon$ [integrating from $0 - i\varepsilon$ to $-i\beta$ is neglected, since $F(z(\tau)) \equiv 0$ for $\tau \in (0 - i\varepsilon, -i\beta)$], that is, integrating over the horizontal part of the contour γ. The function $J(x, y, t; x', y', 0)$ becomes, under this approximation,

$$\begin{aligned} & J(x, y, t; x', y', 0) \\ & = \int \mathscr{D}z(\tau) \exp\left[i \oint d\tau \left[\frac{1}{2} m\dot{z}^2 - V(z) - z(\tau) \sum_{i=1}^{\Omega} f(\varepsilon_i)\, C_{ii} \right] \right. \\ & \quad + \frac{1}{2} \oint d\tau\, d\tau' \Big(F(z(\tau))\, F(z(\tau')) \sum_{i,j=1}^{\Omega} \{\exp[-i(\varepsilon_i - \varepsilon_j)(\tau - \tau')]\} \\ & \quad \left. \times [\theta_\gamma(\tau, \tau') - f(\varepsilon_i)]\, |C_{ij}|^2\, [\theta_\gamma(\tau', \tau) - f(\varepsilon_j)] \Big) \right] \end{aligned} \tag{17}$$

As was used in the original work by Feynman and Vernon,[1] a real-time dynamics of a particle usually is described via two independent functional integrals; one is from time 0 to time t and the other is from time t back to time 0. In parallel to this description, we separate the two branches: If we denote the coordinate variable $z(\tau)$ by $q_1(\tau - i\varepsilon)$ in the upper branch and by $q_2(\tau + i\varepsilon)$ in the lower branch, in which the time arguments of q_1, q_2 are purely real quantities defined in $[0, t]$, then we arrive at the form obtained in Ref. 1,

$$J(x, y, t; x', y', 0) = \int \mathscr{D}q_1\, \mathscr{D}q_2\, F[q_1, q_2] \exp\{iS[q_1] - iS[q_2]\} \tag{18}$$

with the action defined as

$$S[q] \equiv \int_0^t dt' \left[\frac{1}{2} m\dot{q}^2 - V(q) - F(q(t')) \sum_{i=1}^{\Omega} f(\varepsilon_i)\, C_{ii} \right] \tag{18a}$$

and the influence functional given by

$$\begin{aligned} F[q_1, q_2] \equiv \exp\Bigg(-\int_0^t dt_1 \int_0^{t_1} dt_2 \, \{ & [F(q_1(t_1)) - F(q_2(t_1))] \\ & \times \alpha_1(t_1 - t_2)[F(q_1(t_2)) - F(q_2(t_2))] \\ & + i[F(q_1(t_1)) - F(q_2(t_1))]\, \alpha_2(t_1 - t_2)[F(q_1(t_2)) + F(q_2(t_2))] \} \Bigg) \end{aligned} \tag{18b}$$

where

$$\alpha_1(t) = \int_{-\infty}^{+\infty} d\omega\, J_1(\omega) \cos \omega t \tag{19a}$$

$$\alpha_2(t) = \int_{-\infty}^{+\infty} d\omega\, J_2(\omega) \sin \omega t \tag{19b}$$

and

$$J_1(\omega) = \frac{1}{2} \sum_{i,j=1}^{\Omega} |C_{ij}|^2 \frac{e^{\beta\varepsilon_i} + e^{\beta\varepsilon_j}}{(1 + e^{\beta\varepsilon_i})(1 + e^{\beta\varepsilon_j})} \delta(\varepsilon_i - \varepsilon_j - \omega) \tag{20a}$$

$$J_2(\omega) = \frac{1}{2} \sum_{i,j=1}^{\Omega} |C_{ij}|^2 \frac{-e^{\beta\varepsilon_i} + e^{\beta\varepsilon_j}}{(1 + e^{\beta\varepsilon_i})(1 + e^{\beta\varepsilon_j})} \delta(\varepsilon_i - \varepsilon_j - \omega) \tag{20b}$$

No specifications have yet been made with regard to the coefficients C_{ij}. Equations (18)–(20) can be used to study systems in which detailed information about the coefficients could be given via microscopic calculations (in fact, the requirement of the separable interaction condition is unnecessary for this second-order expansion). As an example, let us look at the following simple situation: If near the Fermi surface $\varepsilon = 0$ we have $|C_{ij}|^2 \equiv C^2 = \text{const}$, $\rho(\varepsilon) \equiv \rho_0 \equiv$ density of states at the Fermi energy, then the $J(\omega)$ become

$$J_1(\omega) = \frac{1}{2} \rho_0^2 C^2 \omega \coth(\beta\omega/2) \tag{21a}$$

$$J_2(\omega) = -\frac{1}{2} \rho_0^2 C^2 \omega \tag{21b}$$

Substituting Eqs. (21) back into Eqs. (19) with an expenential cutoff $e^{-\delta|\omega|}$, where $\delta \to 0$, we find

$$\alpha_1(t) = C^2\rho_0^2 \frac{d}{dt}\left[\frac{\pi}{\beta}\coth\frac{\pi}{\beta}(t) - \frac{1}{t} + \frac{t}{t^2+\delta^2} + O\left(\frac{\delta}{\beta}\right)\right]$$

$$\to C^2\rho_0^2 \frac{d}{dt}\left[P\frac{\pi}{\beta}\coth\frac{\pi}{\beta}(t)\right] \tag{19a'}$$

$$\alpha_2(t) = C^2\rho_0^2 \frac{d}{dt}\left(\frac{\delta}{t^2+\delta^2}\right) \to \pi C^2\rho_0^2\delta'(t) \tag{19b'}$$

Thus, the influence functional for this simple coupling becomes

$$F[q_1, q_2] = \exp\left(-\int_0^t dt_1 \int_0^{t_1} dt_2\, C^2\rho_0^2 \left\{[F(q_1(t_1)) - F(q_2(t_1))]\right.\right.$$

$$\times\left[\frac{d}{dt_1}P\frac{\pi}{\beta}\coth\frac{\pi}{\beta}(t_1 - t_2)\right][F(q_1(t_2)) - F(q_2(t_2))]\Big\}$$

$$+\frac{i\pi}{2}\int_0^t dt'\, C^2\rho_0^2\left\{\left[\frac{d}{dt'}F(q_1(t'))\right]F(q_2(t'))\right.$$

$$\left.\left. - F(q_1(t'))\left[\frac{d}{dt'}F(q_2(t'))\right]\right\}\right) \tag{18b'}$$

Meanwhile, there also exists a large adiabatic renormalization potential $V_a(q) = -[C\rho_0 F(q)]^2/\delta$ for the bare system.[3]

These fall into the version of Feynman and Vernon's theory; in particular, Eqs. (21) should be compared to Eqs. (3.10) and (3.11) in Ref. 3 for comparisons between the two kinds of heat baths. The equivalence between the two is thus transparent. In particular, the spectrum is of the Ohmic type of interest in recent research in macroscopic quantum theory.

The approach to Eqs. (18)–(21) differs somewhat from that of Ref. 10. They only kept terms up to second order of the coupling, then brought them back to the exponent without making further justifications. On the other hand, we have essentially included a group of higher order terms up to infinity since our calculation was performed for the exponent. They also took into account part of the interactions between the fermions. But we see no special difficulties in making the same consideration, as mentioned before.

Once the correspondence between the fermionic and bosonic heat baths is found, all work exploring the detailed properties of the bosonic heat bath can also be applied here, including the approach of fixing the

relevant parameters from classical correspondences. Thus, we do not intend to go further along this line; instead, we turn to the search for possible exact solutions for the full Green's function **G** in Eqs. (10)–(12).

4. EXACT SOLUTION IN LONG-TIME LIMIT

Having obtained the second-order expansion for the exponent influence functional, we start to look into the possible exact solution of Eq. (11) in the long-time limit (by long time, we mean long compared to the inverse of the cutoff frequency of the environmental spectrum; see below), and thus give an exact expression for the influence functional in that limit. Equation (11′), when written out in its component form, gives

$$G_{ij} = G^0_{ij}(\tau, \tau') + \sum_{k,l=1}^{\Omega} \oint G^0_{ik}(\tau, \tau'')\, C_{kl} F(z(\tau''))\, G_{lj}(\tau'', \tau')\, d\tau'' \tag{22}$$

What we need is the trace-like function

$$\tilde{G}(\tau, \tau') = \sum_{i,j=1}^{\Omega} G_{ij}(\tau, \tau')\, C_{ji} \tag{23}$$

which satisfies, from Eq. (22),

$$\begin{aligned} \tilde{G}(\tau, \tau') = \tilde{G}_0(\tau, \tau') - i \sum_{i,j,l=1}^{\Omega} \oint d\tau'' \{ &\exp[-i\varepsilon_i(\tau - \tau'')] \} \\ &\times [\theta_\gamma(\tau, \tau'') - f(\varepsilon_i)]\, F(z(\tau''))\, C_{il} C_{ji} G_{lj}(\tau'', \tau') \} \end{aligned} \tag{24}$$

Until now, one can do little with the general Dyson-type equations (22)–(24). To further simplify Eq. (22), we make the following assumptions on C_{ij} in the way of classifying them:

$$\begin{aligned} &C_{ij} = C_\alpha, \qquad \text{for} \quad i, j \in \Omega_\alpha; \qquad\qquad \alpha = 1, \ldots, n \\ &C_{ij} = 0, \qquad \text{for} \quad i \in \Omega_\alpha, \quad j \in \Omega_\beta, \quad \alpha \neq \beta; \quad \beta = 1, \ldots, n \end{aligned}$$

where Ω_α is a subset of the whole state Ω, and $\Omega_1 + \cdots + \Omega_n = \Omega$. Then Eq. (24) reads

$$\begin{aligned} \tilde{G}(\tau, \tau') = \sum_{\alpha=1}^{n} \tilde{G}_\alpha &= \sum_{\alpha=1}^{n} \tilde{G}^0_\alpha(\tau, \tau') \\ &+ \sum_{\alpha=1}^{n} \oint d\tau''\, \tilde{G}^0_\alpha(\tau, \tau'')\, \tilde{G}_\alpha(\tau'', \tau')\, F(z(\tau'')) \end{aligned} \tag{24′}$$

What Eq. (24′) indicates is simply this: Suppose that the Fermi gas has n channels (multichannel environment), such that the interaction matrix coefficients C_{ij} do not connect any two of them; then each of the channels contributes to the influence functional in the same way and the total effect is just the sum of individual contributions (cf. Section 6 for more practical implications). Therefore one only needs to study the case $n=1$ (single-channel environment). We shall return to the multichannel case after finishing the single-channel problem.

Now what is left is a purely mathematical integral equation to be solved. We rewrite Eq. (24′) for the single-channel environment in the form

$$\tilde{G}^0(\tau, \tau') = \tilde{G}^0(\tau, \tau') + \oint d\tau'' \, \tilde{G}^0(\tau, \tau'') \, F(z(\tau'')) \, \tilde{G}(\tau'', \tau') \tag{25}$$

where, from Eqs. (A8) and (23),

$$\tilde{G}^0(\tau, \tau') \equiv -iC \sum_{i=1}^{\Omega} \{\exp[-i\varepsilon_i(\tau - \tau')]\}[\theta_\gamma(\tau, \tau') - f(\varepsilon_i)] \tag{26}$$

Let us examine the detailed structure of $\tilde{G}^0(\tau, \tau')$. To capture the long-time behavior of this function, we choose, for example, the density of states to be $\rho(\varepsilon) = \rho_0 e^{-\delta|\varepsilon|}$ for the moment. Then, in the limit $\delta/\beta \ll 1$, $\beta - |\mathrm{Im}(\tau - \tau')| \gg \delta$,

$$\begin{aligned} \tilde{G}^0(\tau, \tau') &\equiv -iC \sum_{i=1}^{\Omega} \{\exp[-i\varepsilon_i(\tau - \tau')]\}[\theta_\gamma(\tau, \tau') - f(\varepsilon_i)] \\ &\cong -C\rho_0 \frac{\pi/\beta}{\sinh(\pi/\beta)(\tau - \tau' - i\delta \, \mathrm{sgn}_\gamma(\tau, \tau'))} + O\left(\frac{\delta}{\beta}\right) \end{aligned} \tag{26′}$$

where $\mathrm{sgn}_\gamma(\tau, \tau')$ stands for the sign function defined along the contour γ. Detailed plots of Eq. (26′) can be found in Ref. 9, where one can also find some analysis and delicate treatment of the application of the so-called long-time approximation first introduced in Ref. 14. Since the long-time behavior of $\tilde{G}^0(\tau, \tau')$ is quite simple and independent of the cutoff procedure, one might try to approximate Eq. (25) via a replacement of $\tilde{G}^0(\tau, \tau')$ by its long-time form in order to get at least the correst answer for the long-time behavior of $\tilde{G}(\tau, \tau')$. But, because our final goal is to obtain the quantity $\tilde{G}(\tau, \tau^+)$, we have to worry about the contributions from integrals in the short-time regime, since $\tilde{G}^0(\tau, \tau') \propto \delta^{-1}$ for $|\tau - \tau'| \leqslant \delta$; yet it is common knowledge that the adiabatic type of contribution is in general proportional to the number of fermion states involved, namely, $\propto \delta^{-1}$. However, there are no obvious reasons that the two time regimes

will not interfere with each other so that they can be treated separately, as often used previously.[7,9,10]

Based on the above observations, we offer the following procedure to solve Eq. (25) unambiguously to $O(\delta^0)$. To take into account the short-time and long-time regimes consistently, we shall introduce a two-step treatment. In the first step, a delicate long-time truncation of Eq. (25) is considered,

$$\tilde{G}_L^0(\tau, \tau') = \begin{cases} -C\rho_0 P \dfrac{\pi/\beta}{\sinh(\pi/\beta)(\tau - \tau')}, & \tau, \tau' \in \text{same branches} \\ \tilde{G}^0(\tau, \tau'), & \tau, \tau' \notin \text{same branches} \end{cases} \tag{27a}$$

$$\tilde{G}_L(\tau, \tau') = \tilde{G}^0(\tau, \tau') + \oint d\tau'' \, \tilde{G}_L^0(\tau, \tau'') \, F(z(\tau'')) \, \tilde{G}_L(\tau'', \tau') \tag{27b}$$

where P indicates that only the principale part is considered (similar forms of this equation have appeared a number of times in the literature[7,9,10,14,15]). It turns out that Eq. (27b) can be solved exactly if the choice $\rho(\varepsilon) = \rho_0 e^{-\delta|\varepsilon|}$ is employed; the solution is given in Appendix B. Then we go to the remaining part of $\tilde{G}(\tau, \tau')$

$$D(\tau, \tau') = \tilde{G}(\tau, \tau') - \tilde{G}_L(\tau, \tau') \tag{28a}$$

which satisfies

$$D(\tau, \tau') = D^0(\tau, \tau') + \oint d\tau'' \, \tilde{G}^0(\tau, \tau'') \, F(z(\tau'')) \, D(\tau'', \tau') \tag{28b}$$

where

$$D^0(\tau, \tau') = \oint d\tau'' \, [\tilde{G}^0(\tau, \tau'') - \tilde{G}_L^0(\tau, \tau'')] \, F(z(\tau'')) \, \tilde{G}_L(\tau'', \tau') \tag{28c}$$

Equations (28) can be solved up to $O(\delta^0)$ by using the method of Fourier transform. In this way, we thus obtain the solution in the whole time regime to the original integral equation (25). The entire process[8] of solving eq. (25) is presented in Appendix B, including the evaluation of the equal-time limit $\tilde{G}(\tau, \tau^+)$ and related quantities.

[8] We think that a transparent and yet rigorous demonstration of the validity of the long-time plus simple adiabatic approximation used in previous work[7,9,10] is very important, because some questions regarding mathematical rigor have arisen concerning the process of evaluating $\tilde{G}(\tau, \tau^+)$ via this kind of treatment, as the reader can discover. Since such a justification cannot be found in the literature, we give all the details of the solution procedure.

After a long approach, we finally arrive at the goal. In Appendix B, $\tilde{G}(\tau, \tau^+)$ is mainly separated into two parts. The adiabatic part, given by Eq. (B23), contributes an adiabatic potential to the bare system considered. This potential is, if we take most conveniently $\rho(\varepsilon) = \rho_0 \omega_c^2/(\varepsilon^2 + \omega_c^2)$,

$$V_a(z) = \omega_c C \rho_0 \left(\frac{\pi}{2} F(z) - F(z) \arctan[\pi \rho_0 C F(z)] + \frac{\ln\{1 + [\pi \rho_0 C F(z)]^2\}}{2\pi \rho_0 C} \right) \tag{29}$$

The other part, Eqs. (B19) [which is cutoff independent and determined by the long-time behavior of $\tilde{G}^0(\tau, \tau')$] gives the influence functional defined via Eq. (18) in the following form:

$$\begin{aligned} F[q_1, q_2] = \exp\Bigg(& \frac{i}{2\pi} \int_0^t dt' \left\{ \frac{d}{dt'} [Z_1(t')]\, Z_2(t') - Z_1(t') \right. \\ & \left. \times \left[\frac{d}{dt'} Z_2(t') \right] \right\} - \frac{1}{\pi} \int_0^t dt_1 \int_0^{t_1} dt_2 \\ & \times \left\{ [Z_1(t_1) - Z_2(t_1)] \left[\frac{d}{dt_2} \frac{P}{\beta} \coth \frac{\pi}{\beta} (t_2 - t_1) \right] \right. \\ & \left. \times [Z_1(t_2) - Z_2(t_2)] \right\} \Bigg) \end{aligned} \tag{30a}$$

with

$$Z_i(t) \equiv \arctan[\pi \rho_0 C F(q_i(t))], \qquad i = 1, 2 \tag{30b}$$

where we have separated and reorganized the integrals in Eq. (B19a). Equations (30) should be compared to Eq. (18b′).

In the conclusion concerning this single-channel problem, what we have done is to confirm that, based on a rigorous mathematical analysis, apart from the adiabatic potential renormalization of the cutoff dependence (which is of little interest in its detailed structure in the context of macroscopic quantum theory), *the mapping from the fermionic heat bath to the bosonic one via* $\arctan[\pi \rho_0 C F(z)] \to \pi \rho_0 C F(z)$ *holds in general.* This recovers the results of Chang and Chakravarty[9] and Hedegård and Caldeira[10] on their direct considerations of discrete-state systems. However, compared to their results, ours involve much more information concerning the nonlinearities due to the coupling to the environment. This point will be realized in the next two sections.

5. QUANTUM LANGEVIN EQUATION AND CLASSICAL–QUANTUM CORRESPONDENCE

Recall Eq. (24′) of the general multichannel problem; the general density matrix $\rho(x, y, t)$, from Eqs. (5), (18), (29), and (30), can be written in a neat form

$$\begin{aligned}\rho(x, y, t) = \int dx'\, dy'\, \tilde{\rho}(x', y', 0) \\ \times \int \mathscr{D}q_1\, \mathscr{D}q_2 \exp(i\{S[q_1] - S[q_2]\}) \\ \times \exp\left(\frac{i}{2\pi} \sum_{\alpha=1}^{n} \int_0^t dt' \left\{\left[\frac{d}{dt'} Z_1^\alpha(t')\right] Z_2^\alpha(t') \right.\right. \\ \left.\left. - Z_1^\alpha(t')\left[\frac{d}{dt'} Z_2^\alpha(t')\right]\right\} \right. \\ - \frac{1}{\pi^2} \sum_{\alpha=1}^{n} \int_0^t dt_1 \int_0^{t_1} dt_2\, [Z_1^\alpha(t_1) - Z_2^\alpha(t_1)] \\ \left. \times \left[\frac{d}{dt_2} P\left(\frac{\pi}{\beta}\right) \coth \frac{\pi}{\beta}(t_2 - t_1)\right] [Z_1^\alpha(t_2) - Z_2^\alpha(t_2)]\right) \end{aligned} \tag{31}$$

with

$$Z_i^\alpha(t) \equiv \arctan[\pi\rho_0 C_\alpha F(q_i(t))], \qquad i = 1, 2; \quad \alpha = 1, \ldots, n \tag{32}$$

where $F(q)$ is the interaction function to be determined in practical applications, as mentioned above, and all the potential renormalization terms should be absorbed into the actions $S[q_1]$ and $S[q_2]$. The expression (31), applicable to both classical and quantum regimes, containing a considerable amount of nonlinearity, has a large range of applications. Details are to be explored.

In order to fix the function $F(q)$ and the coefficients C_α, we consider an important and practically interesting regime—the so-called quasiclassical regime. In this case, a very similar argument can be used to that of Schmid[4] for the quasiclassical Langevin equation in a bosonic heat bath.[9] An outline of the procedure is as follows: We first change the double integral term in Eq. (31) into a Gaussian stochastic term contributing to an effective action $S_{\text{eff}}[q]$. Then we transform q_1, q_2 into

$$Q = (q_1 + q_2)/2, \qquad r = q_1 - q_2$$

[9] Detailed requirements for this kind of approximation can be found in Ref. 4. There is a similar equation in Ref. 5, dealing with an electron bath in a Josephson tunnel system.

After this we argue that the stationary phase gives the most important contributions to the density matrix. Upon doing this, a quasiclassical equation of motion is found by requiring $(\delta/\delta r)\, S_{\text{eff}}[Q, r] = 0$. To actually pursue these steps, we have

$$\exp\left\{-\frac{1}{2\pi^2}\sum_{\alpha=1}^{n}\int_0^t dt_1 \int_0^{t_1}[Z_1^\alpha(t_1) - Z_2^\alpha(t_1)]\right.$$
$$\left.\times\left[\frac{d}{dt_2}P\left(\frac{\pi}{\beta}\right)\coth\frac{\pi}{\beta}(t_2 - t_1)\right][Z_1^\alpha(t_2) - Z_2^\alpha(t_2)]\right\}$$
$$= \left\langle \exp\left\{i\sum_{\alpha=1}^{n}\int_0^t dt\, f(t)[Z_1^\alpha(t) - Z_2^\alpha(t)]\right\}\right\rangle \tag{33}$$

where $f(t)$ is a random Gaussian stochastic force satisfying

$$\langle f(t)\, f(t')\rangle = \frac{1}{\pi^2}\left[\frac{d}{dt'}P\left(\frac{\pi}{\beta}\right)\coth\frac{\pi}{\beta}(t' - t)\right] \tag{34}$$

In this way, we can define a total effective action as

$$S_{\text{eff}}[Q, r] = \int_0^t dt\left(m\dot{r}(t)\,\dot{Q}(t) - V_r\left(Q + \frac{r}{2}\right) + V_r\left(Q - \frac{r}{2}\right)\right.$$
$$+ f(t)\sum_{\alpha=1}^{n}\left[\bar{Z}_\alpha\left(Q + \frac{r}{2}\right)\right.$$
$$\left.\left. - \bar{Z}_\alpha\left(Q - \frac{r}{2}\right)\right] + \frac{1}{2\pi}\sum_{\alpha=1}^{n}\left\{\left[\frac{d}{dt}\bar{Z}_\alpha\left(Q + \frac{r}{2}\right)\right]\bar{Z}_\alpha\left(Q - \frac{r}{2}\right)\right.\right.$$
$$\left.\left.\left. - \bar{Z}_\alpha\left(Q + \frac{r}{2}\right)\left[\frac{d}{dt}\bar{Z}_\alpha\left(Q - \frac{r}{2}\right)\right]\right\}\right) \tag{35}$$

with the definition of the function $\bar{Z}_\alpha(Q)$,

$$\bar{Z}_\alpha(Q) \equiv \arctan[\pi\rho_0 C_\alpha F(Q)] \tag{36}$$

This serves to give the Langevin equation for the system by

$$\frac{\delta}{\delta r(t)} S_{\text{eff}}[Q, r]\Big|_{r=0}$$
$$= -m\ddot{Q} - V'(Q) + \sum_{\alpha=1}^{n}\left[f(t)\,\bar{Z}'_\alpha(Q) - \frac{\bar{Z}'^2_\alpha(Q)}{\pi}\dot{Q}(t)\right] = 0 \tag{37}$$

the significance of which is discussed below.

Next, we make a very simple application of the obtained quantum Langevin equation (37). Suppose that one is not interested in the dynamics of the particle itself, but instead in the energy absorption capacity of the fermionic environment. An easy way to look for the answer is to let the particle be so heavy that its motion is not affected by the presence of the environment. Thus, the energy absorbed by the environment from the particle is simply

$$E(0, t)=\frac{1}{\pi}\sum_{\alpha=1}^{n}\int_0^t dt\,\dot{\bar{Z}}_\alpha^2=\frac{1}{\pi}\sum_{\alpha=1}^{n}\int_{\bar{Z}_\alpha(0)}^{\bar{Z}_\alpha(t)}\dot{\bar{Z}}_\alpha\,d\bar{Z}_\alpha \tag{38}$$

The function $\bar{Z}_\alpha(Q)$ here is in fact quite arbitrary. To be specific, let us first take $\bar{Z}_\alpha(Q)$ to be $\arctan(\pi\rho_0 C_\alpha Q)$ and consider a particle moving with a velocity v_0 arriving at $Q=0$ at time $t=0$. What Eq. (38) says is

$$E(-\infty, +\infty)=\frac{1}{2}\sum_{\alpha=1}^{n}|\pi\rho_0 C_\alpha v_0| \tag{39}$$

An interesting phenomenon is thus found: the energy dissipated into the environment is saturated. In fact, the energy is mainly dissipated in the region $|\pi\rho_0 C_\alpha Q|\leqslant 1$. However, one might say that this is quite artificial; so we look at a more realistic situation. Suppose that $F(Q)$ is a periodic function of Q; then, because of the presence of the arctan function inside the $\bar{Z}_\alpha(Q)$, Eq. (38) has the following significance: As the amplitude of $F(Q)$ increases, the energy dissipated per cycle will tend to saturate at a finite value, which is nearly the quantity $2\times\frac{1}{2}\sum_{\alpha=1}^{n}|\pi\rho_0 C_\alpha \dot{F}|_{F=0}$.

The quantum Langevin equation (38) gives, among other things, a general way to determine the interaction function $F(Q)$ and the coupling coefficients C_α from the corresponding classical equation of motion in analogy to what has been done for the boson bath coupling.[3] Thus, the whole procedure of modeling the fermionic environment is self-consistent and is independent of the previous work on the boson bath coupling. Nevertheless, there are close relations between the two cases. As we have seen, the mapping from the single-channel fermion bath to the Ohmic boson bath can be simply achieved: If a particle interacts with a fermion bath via an interaction function $\pi\rho_0 CF(z)$, then the effects of the environment are identical to those of a boson bath with a corresponding interaction function $\arctan[\pi\rho_0 CF(z)]$. In this situation, one *cannot* distinguish the two baths if the information concerning the quantum mechanical description comes only from the classical equation of motion. Thus it is perfectly alright to employ the commonly used boson bath with a separable

coupling Hamiltonian in the context of macroscopic quantum theory.[2] But, one ought to keep in mind that the boson bath transformed from the fermion bath has a saturation property in the effective interaction between the system and the environment, while an ordinary boson bath does not. As a result, the energy absorption by the fermionic environment can be limited. With regard to the multichannel case, the situation is somewhat complicated. The correspondence between a multichannel fermionic environment and an environment consisting of many boson baths can be made to complete the mapping, although this is somewhat artificial.

6. COMMENTS ON GENERALIZATIONS, APPLICATIONS, AND CONNECTIONS TO OTHER WORKS

6.1. Generalizations and Applications

The multichannel model that we have considered contains a considerable amount of information about the environment. Nevertheless, in real life the environment might not fall into the simple model. But in certain circumstances, our model does give some hints to actual problems. For example, if the environment can be shown to be basically a multichannel one plus minor corrections on the matrix elements connecting different channels, then we expect that a simple perturbation treatment on the multichannel basis can be acceptable. The generalization to the case of more than one species of Fermi gas such that each different species has its own interaction function $F(q)$ is obvious. In all of these situations, the detailed structure of the environments should be known to a certain extent. Furthermore, as a complement to the quantum Langevin equation approach, a force equation with a time-dependent external force can be obtained via our complex-time path integral representation, in analogy to that in Ref. 25.

To get nontrivial generalizations, let us return to the general cases where $C_{ij}(z)$ cannot be simply separated into $C_{ij}F(z)$. However, there exists a very important class where the scattering matrix elements $C_{ij}(z)$ do not involve explicitly energy dependences over a certain range of interest, namely, they are independent of $\{\varepsilon_i, \varepsilon_j\}$. Then at least part of the problem can be nearly reduced to the simple multichannel type discussed above. For definiteness, let us consider a specific example, such as coupling to an

[10] This orthogonal expansion is not unique; other expansions can always be made, according to convenience. For instance, in some cases where the explicit direction of the coordinate z might be involved, different choices may be made, depending on the specific features of $C_{ij}(z)$.

isotropic Fermi liquid environment. In this example, we can simply expand[10] (spin complications are not discussed here, for simplicity)

$$C_{ij}(z) \to C_{\mathbf{p},\mathbf{p}'}(z) = \sum_{l=0}^{+\infty} C_l(z)\, P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') \tag{40a}$$

$$G_{ij}(\tau, \tau') \to G_{\mathbf{p},\mathbf{p}'}(\tau, \tau') = \sum_{l=0}^{+\infty} G_l(\tau, \tau'; \varepsilon_p, \varepsilon_{p'})\, P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') \tag{40b}$$

in terms of Legendre polynomials, where all the notations are conventional. Now, recalling Eqs. (23) and (24), we can easily derive the following integral equation:

$$\begin{aligned} &\sum_{l=0}^{+\infty} \frac{C_l(z(\tau))}{2l+1} \tilde{G}_l(\tau, \tau') \\ &\quad = \sum_{l=0}^{+\infty} \Bigg[C_l(z(\tau))\, \tilde{G}^0(\tau, \tau') \\ &\qquad + \frac{C_l(z(\tau))}{(2l+1)^2} \oint d\tau'' \tilde{G}^0(\tau, \tau'')\, C_l(z(\tau''))\, \tilde{G}_l(\tau'', \tau') \Bigg] \end{aligned} \tag{41}$$

with the definitions

$$\tilde{G}_l(\tau, \tau') = \int_{-\infty}^{+\infty} d\varepsilon_p\, d\varepsilon_{p'}\, \rho(\varepsilon_p)\, \rho(\varepsilon_{p'})\, \tilde{G}_l(\tau, \tau'; \varepsilon_p, \varepsilon_{p'}) \tag{42a}$$

$$\tilde{G}^0(\tau, \tau') = \sum_{\mathbf{p}} G^0_{\mathbf{p},\mathbf{p}}(\tau, \tau') \tag{42b}$$

Equation (41) is already in the form of the multichannel problem (24′) with only trivial modifications. The changes in the final expression for the density matrix are the following simple replacements in Eqs. (31) and (32):

$$C_\alpha F(z) \to \frac{C_l(z)}{(2l+1)}$$

and an overall factor $(2l+1)$ for the lth branch; equivalently, apart from the adiabatic potential renormalizations,

$$\arctan[\pi\rho_0 C_\alpha F(z)] \to (2l+1)^{1/2} \arctan\left[\frac{\pi\rho_0 C_l(z)}{(2l+1)}\right]$$

In fact, this sort of generalization covers a variety of practical interests. In addition, in the second order expansion (17), arbitrary $C_{ij}(z)$ can be used.

This result has very important implications in the process of two-state dynamics[6,9], where the effective coupling strength plays an essential role. For the spin-boson problem, it is generally believed that if the dimensionless coupling constant $\alpha > 1$ (cf. Ref. 6) the dynamics will result in a spontaneous symmetry breaking, leading to the localization of the system inside one of the two states. For a spin-fermion problem, previous calculations[9,10] showed that, when converted into the spin-boson problem, the system has an equivalent coupling constant never greater than one. Researchers have thus concluded that a spin-fermion system will probably not encounter symmetry breaking. However, our simple generalized result changes this conclusion. As the coupling between the system and the fermionic environment becomes complicated and involves many branches of the fermion bath [see Eq. (40)], the upper limit of the equivalent coupling constant is no longer restricted to 1, especially for branches with large l (cf. also the comment at the end of the last section).

Although our real-time approach is physically transparent and contains most of the essential information about the dynamics of the quantum system, it is not quite known how to use this path integral formalism to evaluate the MQT rate out of a metastable potential. To solve this difficulty, calculations of the imaginary part of the ground-state energy or free energy were performed to find the tunneling rate,[2,22-24] in which a so-called instanton technique borrowed from quantum field theory was commonly used. In the procedure, the effective Euclidean action for the system alone played an essential role. For the sake of self-completeness, we also present the corresponding Euclidean action for the fermionic environment, which was worked out previously by Hamann[7] in a slightly different version. A brief derivation of the action is carried out in Appendix C. It turns out that [recall Eqs. (27) and (28a)]

$$S_{\text{eff}}[z(\tau)] = \int_0^\beta d\tau \left[\frac{1}{2} m\dot{z}^2 + V(z) + V_a(z) \right] + \frac{1}{4\pi^2} \int_0^\beta d\tau \, d\tau' \left(\frac{\pi}{\beta} \right)^2 \left[\frac{Z(\tau) - Z(\tau')}{\sin \pi/\beta(\tau - \tau')} \right]^2 \tag{43}$$

This serves as confirmation of the above real-time results.

6.2. Connections to Other Works

Those working on macroscopic quantum theory are particularly interested in the Josephson tunnel system—in fact, this is the most suitable physical system for experimental test of the theory. A microscopic confirmation with regard to the validity of taking the phase variable as a coor-

dinate-like variable was given by Ambegaokar *et al.*[5] In their work, one can find an interesting result: the dissipation of the Josephson tunnel junction is due to couplings to two independent environments via $\cos\frac{1}{2}\phi$ and $\sin\frac{1}{2}\phi$, respectively. Furthermore, one finds that their calculation is essentially a second-order result in a functional integral approach, with respect to the tunneling matrix of the electrons between the two superconductors. These, however, falls into our formalism for coupling to two independent species of Fermi gases with interaction function $F(q)$ chosen to be $\cos\frac{1}{2}q$ and $\sin\frac{1}{2}q$, respectively. Thus significant connections between these two approaches should be sought.

Some workers have already looked into possible Bloch-wave-like phenomena in a Josephson junction[18] in the limit of a vanishingly small shunt capacitance, and thus possibly large relative coupling to the environment (the typical dynamic frequency of the order of the Josepson frequency ω_J can be larger than the gap frequency $2\Delta/\hbar$, leading to an increase of the quasiparticle scattering). Then, through our formalism, significant higher order corrections of a purely fermionic nature could be quite crucial. Details will be considered later.

When one actually considers the Bownian motion of a particle moving in a Fermi liquid, for example, a heavy ion in a ^{3}He liquid, it is more plausible that the coupling to the Fermi gas would be via the momentum of the particle rather than the coordinate of the particle after the adiabatic potential part is drawn out[4,19]. Thus, one may ask whether it is possible to transform the momentum-like coupling into a coordinate-like one. This is easily done for a boson bath provided the coupling is linear.[20] However, it is still a challenge in the fermionic case. We intend to look into details of this question.

A simple property of Eq. (37) is that the energy absorption of the environment tends to be saturated as the coupling strength C increases, or, equivalently, the amplitude of the disturbing force increases, as is shown in Eqs. (38) and (39). Nevertheless, as the number of channels increases, the saturation disappears. We suspect that such a quantum saturation phenomenon may be actually observed in a mixture of ^{3}He and ^{4}He liquids with a low concentration of ^{3}He when it is disturbed with an extremely thin lead. Detailed conditions and analyses remain to be explored.

We mention that there exist another approach to the bosonization of a fermionic bath, which is to work directly on the possible operator transformations between the two baths (see Ref. 21 and references therein). Here we emphasize that to show the equivalence between the two baths, one must also show that the ground states of the two systems can also be transformed into each other under the same operator transformations! In fact, this is not an easy approach either.

Finally, we discuss briefly the possibility of using the "right variable" perturbation to deal with this fermion bath problem. For the spin-boson problem[6] there exists an alternative approach[26] where one starts from a "displaced" Hamiltonian (namely, the harmonic oscillators of the boson bath are displaced to new stable positions under a suitable unitary transformation), then performs perturbation iterations on the real-time equations of motion. The results are largely in agreement with those of the Feynman—Vernon technique.[6] Therefore, one may wonder whether a similar method can be applied here. However, we find that in spite of the difficulties there are, in fact, several reasons to actually pursue this ideal; they are clustered into the following three major points:

(1) The conclusion that for the spin-boson problem the path integral results can be recovered by those of the right second-order pertubation has limitations. Prior to the path integral treatment[6] by Leggett *et al.*, a number of fine results for specific regions of the parameter space could be found in the literature. One of the sucesses of the Feynman–Vernon technique was the elimination of the boundaries between the various regions. In addition, besides the nice analytic results under the celebrated "noninteration-blip approximation," the treatment gave several series expansions which are well-controlled for numerical evaluations. On the other hand, the results of the second-order perturbation in Ref. 26 only agree with the noninteraction-blip results (note that there are differences in the parameter ranges of validity), but leave the remaining iteration scheme quite uncontrolled. After all, these are different schemes in solving the spin-boson problem, and all involve certain approximations. Thus they are, in some sense, irrelevant to the goal of "exactly theoretical formalism" in this work (cf. below).

(2) For the spin-boson problem it is trivial to find the displaced Hamiltonian. However, the corresponding procedure for the spin-fermion system is a completely different story. The coupling now involves two anticommutation fermion operators. One possible scheme is to first transform the fermion field into a boson field. But, as was mentioned above, one must also transform the old ground state into the new one (for this reason the result in Ref. 21, we suspect, is unreliable). Of course, for the second order of the coupling it is entirely trivial to make the above transformation. Beyond this approximation, to the best of our knowledge, no simple scheme has yet been found (especially for a extended-coordinate system).

(3) Finally, since the equivalence between spin-boson and spin-fermion problems was found (at least partly) by Chang and Chakravarty in Ref. 9, the goal of this work is to propose a unified path integral formalism for the study of fermionic bath problems, with particular emphasis on the

general extended quantum system. Quite naturally the result of Chang and Chakravarty is only a specific case in this formalism. What is more important is that the new formalism shows the complete nonlinearities of the fermion bath problem, which were little-studied previously.

Nevertheless, it is still an interesting problem whether a much simpler description could arrive at the same goal (we strongly suspect that it indeed exists). To date, it is unclear.

APPENDIX A. DERIVATION OF THE PATH INTEGRAL REPRESENTATION (7)

In this appendix, we express Eq. (6) in the text via a path integral representation. First, let us separate the Hamiltonian (1) into

$$\hat{H}(\tau) = \hat{H}_p(\tau) + \hat{H}_e^0(\tau) + \hat{H}_{e-p}(\hat{x}, \{\hat{b}_i^\dagger, \hat{b}_i\}) + \hat{H}_{\text{int}}(\{\hat{b}_i^\dagger, \hat{b}_i\}) \tag{A1a}$$

$$\hat{H}_0(\tau) = \hat{H}_p(\tau) + \hat{H}_e^0(\tau) \tag{A1b}$$

where each part has its own physical meaning:

$$\hat{H}_p(\tau) \equiv \frac{\hat{p}^2}{2m} + V(\hat{x}) + \eta(\tau)\,\hat{x} \tag{A2a}$$

$$\hat{H}_e^0 \equiv \sum_{i=1}^{\Omega} \hat{b}_i^\dagger \varepsilon_i \hat{b}_i + \sum_{i=1}^{\Omega} [\lambda_i^\dagger(\tau)\,\hat{b}_i + \hat{b}_i^\dagger \lambda_i(\tau)] \tag{A2b}$$

$$\hat{H}_{e-p}(\hat{x}, \{\hat{b}_i^\dagger, \hat{b}_i\}) \equiv \sum_{i,j=1}^{\Omega} C_{ij}(\hat{x})\,\hat{b}_i^\dagger \hat{b}_j \tag{A2c}$$

In Eqs. (A2), we have deliberately introduced some external source terms, in which $\eta(\tau)$ is an ordinary τ-*dependent* function and $\{\lambda_i^\dagger(\tau), \lambda_i(\tau)\}$ is a set of functions obeying the Grassmann algebra.[11] With this introduction of the sources, we are able to draw out all the interactions easily by

$$\begin{aligned} &J(x, y, t; x', y', 0) \\ &\equiv \mathrm{Tr}_e\left[T_\gamma \left(\exp\left\{ -i \int_{0-i\varepsilon}^{-i\beta} [\hat{H}_e^0(\tau) + \hat{H}_{\text{int}}(\tau)]\, d\tau \right\} \right.\right. \\ &\quad \times \langle y' | \exp\left[-i \int_{t-i\varepsilon}^{0-i\varepsilon} \hat{H}(\tau)\, d\tau \right] | y \rangle \\ &\quad \left.\left. \times \langle x | \exp\left[-i \int_{0+i\varepsilon}^{t+i\varepsilon} \hat{H}(\tau)\, d\tau \right] | x' \rangle \right) \right]\Bigg|_{\eta(\tau), \lambda_i^\dagger(\tau), \lambda_i(\tau), \varepsilon \to 0} \end{aligned}$$

[11] This introduction is, in fact, a common method used in functional approach to quantum field theory, see ref. 11, 17.

$$
\begin{aligned}
&\equiv \exp\left(-i\int_\gamma d\tau\left[\hat{H}_{e-p}\left(-\frac{\delta}{i\delta\eta(\tau)},\left\{\frac{\delta}{i\delta\lambda_i(\tau)},-\frac{\delta}{i\delta\lambda_i^\dagger(\tau)}\right\}\right)\right.\right.\\
&\quad\left.\left.+\hat{H}_{\text{int}}\left(\left\{\frac{\delta}{i\delta\lambda_i(\tau)},-\frac{\delta}{i\delta\lambda_i^\dagger(\tau)}\right\}\right)\right]\right)\\
&\quad\times \mathrm{Tr}_e\left(T_\gamma\left\{\exp\left[-i\int_{0-i\varepsilon}^{-i\beta}\hat{H}_e^0(\tau)\,d\tau\right]\right.\right.\\
&\quad\times\langle y'|\exp\left[-i\int_{t-i\varepsilon}^{0-i\varepsilon}\hat{H}_0(\tau)\,d\tau\right]|y\rangle\\
&\quad\left.\left.\times\langle x|\exp\left[-i\int_{0+i\varepsilon}^{t+i\varepsilon}\hat{H}_0(\tau)\,d\tau\right]|x'\rangle\right\}\right)\Bigg|_{\eta(\tau),\lambda_i^\dagger(\tau),\lambda_i(\tau),\varepsilon\to 0}\\
&\equiv \exp\left(-i\int_\gamma d\tau\left[\hat{H}_{e-p}\left(-\frac{\delta}{i\delta\eta(\tau)},\left\{\frac{\delta}{i\delta\lambda_i(\tau)},-\frac{\delta}{i\delta\lambda_i^\dagger(\tau)}\right\}\right)\right.\right.\\
&\quad\left.\left.+\hat{H}_{\text{int}}\left(\left\{\frac{\delta}{i\delta\lambda_i(\tau)},-\frac{\delta}{i\delta\lambda_i^\dagger(\tau)}\right\}\right)\right]\right)\\
&\quad\times \mathrm{Tr}_e\left(T_\gamma\exp\left[-i\int_\gamma\hat{H}_e^0(\tau)\,d\tau\right]\right)\\
&\quad\times\langle y'|\exp\left[-i\int_{t-i\varepsilon}^{0-i\varepsilon}\hat{H}_p(\tau)\,d\tau\right]|y\rangle\\
&\quad\left.\times\langle x|\exp\left[-i\int_{0+i\varepsilon}^{t+i\varepsilon}\hat{H}_p(\tau)\,d\tau\right]|x'\rangle\right)\Bigg|_{\eta(\tau),\lambda_i^\dagger(\tau),\lambda_i(\tau),\varepsilon\to 0}
\end{aligned}
\tag{A3}
$$

where the contour γ is described in Fig. 1, and T_γ stands as the time order operator with respect to the contour. Equation (A3) is a common technique used in field theory to separate two interacting systems.

Now we look into some of well known path integral formulas. A fundamental formula[16] in path integral approach to quantum mechanics is

$$
\begin{aligned}
&\langle x|\exp\left\{-i\int_0^t\left[\frac{\hat{p}^2}{2m}+V(\hat{x})+\eta(\tau)\,\hat{x}\right]d\tau\right\}|x'\rangle\\
&\quad\equiv\int_{z(0)=x'}^{z(t)=x}\mathscr{D}z(\tau)\exp\left\{i\int_0^t d\tau\left[\frac{1}{2}m\left(\frac{dz}{d\tau}\right)^2-V(z)-\eta(\tau)\,z(\tau)\right]\right\}
\end{aligned}
\tag{A4}
$$

This enables us to cast the system part in Eq. (A3) into path integral form. As regards the environmental part, we have

$$\mathrm{Tr}_e\left(T_\gamma \exp\left[-i\int_\gamma \hat{H}_e^0(\tau)\,d\tau\right]\right)$$
$$\equiv \mathrm{Tr}_e\left(T_\gamma \exp\left\{-i\int_\gamma d\tau \sum_{i=1}^{\Omega}\left[\hat{b}_i^\dagger \varepsilon_i \hat{b}_i + \lambda_i^\dagger(\tau)\,\hat{b}_i + \hat{b}_i^\dagger \lambda_i(\tau)\right]\right\}\right)$$
$$\equiv \mathrm{Tr}_e\left(\left[\exp(-\beta \hat{H}_e^0)\right] T_\gamma \exp\left\{-i\int_\gamma d\tau \sum_{i=1}^{\Omega}\left[\hat{b}_i^\dagger(\tau)\,\lambda_i(\tau) + \lambda_i^\dagger(\tau)\,\hat{b}_i(\tau)\right]\right\}\right) \tag{A5}$$

where the time-dependent operators $\hat{b}_i^\dagger(\tau)$ and $\hat{b}_i(\tau)$ are defined as

$$\hat{b}_i^\dagger(\tau) \equiv \exp(i\hat{H}_e^0\tau)\,\hat{b}_i^\dagger \exp(-i\hat{H}_e^0\tau) \qquad \tau \in \gamma \tag{A6a}$$

$$\hat{b}_i(\tau) \equiv \exp(i\hat{H}_e^0\tau)\,\hat{b}_i \exp(-i\hat{H}_e^0\tau) \qquad \tau \in \gamma \tag{A6b}$$

with

$$\hat{H}_e^0 \equiv \sum_{i,j=1}^{\Omega} \hat{b}_i^\dagger \varepsilon_i \hat{b}_i \tag{A6c}$$

After applying Wick's theorem[11,17] to Eq. (A5), we get

$$\mathrm{Tr}_e\left(T_\gamma \exp\left[-i\int_\gamma \hat{H}_e^0(\tau)\,d\tau\right]\right)$$
$$\equiv \mathrm{Tr}_e\left[\exp\left(-\beta \hat{H}_e^0\right)\right] \exp\left\{-i\int_\gamma d\tau\,d\tau' \sum_{i,j=1}^{\Omega} \lambda_i^\dagger(\tau)\,G_{ij}^0(\tau,\tau')\,\lambda_j(\tau')\right\} \tag{A7}$$

where $G_{ij}^0(\tau,\tau')$ refers to the one-particle Green's function[9] with the definition

$$G_{ij}^0(\tau,\tau') = -i\frac{\mathrm{Tr}\{[\exp(-\beta \hat{H}_e^0)]\,T_\gamma[\hat{b}_i(\tau)\,\hat{b}_j^\dagger(\tau')]\}}{\mathrm{Tr}[\exp(-\beta \hat{H}_e^0)]}$$
$$= -i\delta_{ij}\{\exp[-i\varepsilon_i(\tau-\tau')]\}[\theta_\gamma(\tau,\tau') - f(\varepsilon_i)] \tag{A8}$$

and $\theta_\gamma(\tau,\tau')$ in Eq. (A8) is the step function defined along the contour γ. However, if we look at the following path integral in Grassmann algebra,[11]

$$\int_{b_i(0+i\varepsilon)=-b_i(-i\beta)}^{b_i^\dagger(0+i\varepsilon)=-b_i^\dagger(-i\beta)} \prod_{i=1}^{\Omega} \mathscr{D}b_i^\dagger(\tau)\,\mathscr{D}b_i(\tau)$$
$$\times \exp\left\{i\int_\gamma d\tau \sum_{i=1}^{\Omega}\left[b_i^\dagger(i\partial_\tau - \varepsilon_i)\,b_i(\tau) - \lambda_i^\dagger(\tau)\,b_i(\tau) - b_i^\dagger(\tau)\,\lambda_i(\tau)\right]\right\}$$
$$\equiv \text{const} \times \exp\left[-i\int_\gamma d\tau\,d\tau' \sum_{i,j=1}^{\Omega} \lambda_i^\dagger(\tau)\,M_{ij}(\tau,\tau')\,\lambda_i(\tau)\right] \tag{A9}$$

where $M_{ij}(\tau, \tau')$ satisfies

$$(i\partial_\tau - \varepsilon_i)\, M_{ij}(\tau, \tau') = \delta_{ij}\, \delta_\gamma(\tau, \tau') \tag{A10a}$$

with the boundary conditions

$$M_{ij}(0 + i\varepsilon, \tau') = -M_{ij}(-i\beta, \tau') \tag{A10b}$$

$$M_{ij}(\tau, 0 + i\varepsilon) = -M_{ij}(\tau, -i\beta) \tag{A10c}$$

Equations (A10) are enough to determine $M_{ij}(\tau, \tau')$. It turns out to be $G^0_{ij}(\tau, \tau')$ given in Eq. (A8), so that the left sides of Eqs. (A7) and (A9) coincide (identities containing path integrals are made up to an irrelevant normalization factor). Thus, we are now ready to express the function $J(x, y, t; x', y', 0)$ in terms of the path integral. Combining Eq. (A3) with Eqs. (A4), (A7), and (A9) and restoring all the interactions, we finally arrive at

$$\begin{aligned} &J(x, y, t; x', y', 0) \\ &\quad = \int \mathscr{D}z(\tau) \prod_{i=1}^{\Omega} \mathscr{D}b_i^\dagger(\tau)\, \mathscr{D}b_i(\tau) \exp\left(i \int_\gamma d\tau \left[\frac{1}{2} m \left(\frac{dz}{d\tau} \right)^2 \right.\right. \\ &\qquad \left.\left. - V(z) - \sum_{i,j=1}^{\Omega} C_{ij}(z)\, b_i^\dagger b_j + \sum_{i=1}^{\Omega} b_i^\dagger (i\partial_\tau - \varepsilon_i)\, b_i - H_{\text{int}}(\{b_i^\dagger, b_i\}) \right] \right) \end{aligned} \tag{A11}$$

This is nothing but Eq. (7) in Section 2. The boundary conditions associated with it are also stated there.

APPENDIX B. SOLVING THE SINGULAR INTEGRAL EQUATION (25)

B.1. Solution of Equation (27)

As mentioned in the text, we shall solve this integral equation in two steps. In the first step, we concentrate on Eq. (27) to capture all the essential long-time properties. Here, for definiteness, we choose the density of states of the environment to be $\rho(\varepsilon) = \rho_0 e^{-\delta|\varepsilon|}$ throughout the two steps. At the end of this appendix we return to the general situation of arbitrary choices on the high-energy cutoff. For later convenience in keeping track of $\tilde{G}^0_L(\tau, \tau')$ easily, let us make a simple contour deformation[9]: Shift the upper (lower) branch of the contour γ (i.e., of the integral loop $\oint$) to $[0 + i\delta/2, t + i\delta/2]$ ($[t - i\delta/2, 0 - i\delta/2]$) and denote this new contour by L; see Fig. 2. As a consequence, we have $\oint \to \int_L$. This deformation is perfectly

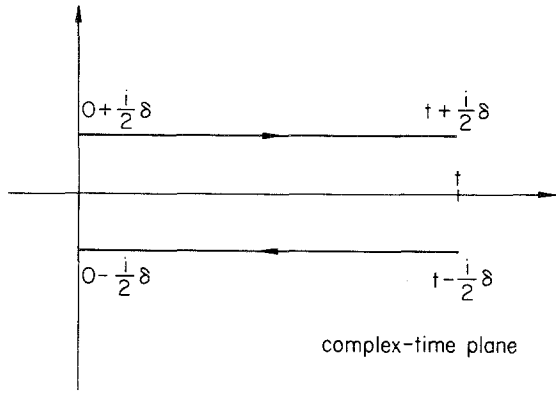

Fig. 2. The contour L. It consists of two straight lines with definite directions. The two lines are separated by δ, which is a small, but finite quantity.

legitimate, since originally all the functions concerned in Eqs. (25)–(28) were only defined on the contour γ, so that this produces only slight changes in the notation of the argument-function correspondences. Namely, we now define the *same* functions using the corresponding time arguments on L. This greatly simplifies the expression for Eq. (26b); it is simply

$$\psi(\tau, \tau) - \frac{P}{\pi i} \int_L d\tau'' \, B(\tau'') \, \psi(\tau'', \tau') \frac{\pi/\beta}{\sinh(\pi/\beta)(\tau'' - \tau)} = \psi^0(\tau, \tau') \qquad \text{(B1)}$$

where we have scaled the quantities

$$\psi(\tau, \tau') \to \frac{\tilde{G}_L(\tau, \tau')}{C\rho_0}, \qquad \psi^0(\tau, \tau') \to \frac{\tilde{G}^0_L(\tau, \tau')}{C\rho_0}, \qquad B(\tau) \to i\pi\rho_0 C F(z(\tau))$$

and the integral contour is defined in Fig. 2 (note that we do not actually enclose the contour).

We shall solve this singular integral equation by the method introduced in Muskhelishvili's book.[15] Equation (B1) is not yet in the form of the Hilbert problems discussed in Ref. 15. However, if we use the identities[12]

$$\frac{\pi/\beta}{\sinh(\pi/\beta)(\tau - z)} \equiv \sum_{-\infty}^{+\infty} \frac{(-1)^n}{\tau - z + in\beta} \qquad \text{(B2a)}$$

$$\frac{\pi}{\beta} \coth \frac{\pi}{\beta}(\tau - z) \equiv \sum_{-\infty}^{+\infty} \frac{1}{\tau - z + in\beta} \qquad \text{(B2b)}$$

[12] We also use the symbol z to denote an arbitrary complex time variable in the process of solving Eq. (B1). This should not be confused with the coordinate variable used in our path integrals.

and the antiperiodicity of $\psi(\tau, \tau')$,

$$\psi(\tau + in\beta, \tau') = (-1)^n \psi(\tau, \tau') \tag{B2c}$$

$$\psi(\tau, \tau' + in\beta) = (-1)^n \psi(\tau, \tau') \tag{B2d}$$

we can cast Eq. (B1) into the standard form of Hilbert problems with a new contour $L' \equiv L + in\beta$, $n = -\infty$ to $+\infty$, as shown in Fig. 3,

$$\psi(\tau, \tau') - \frac{P}{\pi i} \int_{L'} d\tau'' \frac{B(\tau'')\, \psi(\tau'', \tau')}{\tau'' - \tau} = \psi^0(\tau, \tau') \tag{B3}$$

where $B(\tau'' + in\beta)$ is assumed to be $B(\tau'')$. The solution to Eq. (B3) is well constructed in Ref. 15. We briefly repeat the procedure here.

Define a sectional holomorphic function as in Ref. 15,

$$\Phi(z) = \frac{1}{2\pi i} \int_{L'} d\tau'' \frac{B(\tau'')\, \psi(\tau'', \tau')}{\tau'' - z} \tag{B4}$$

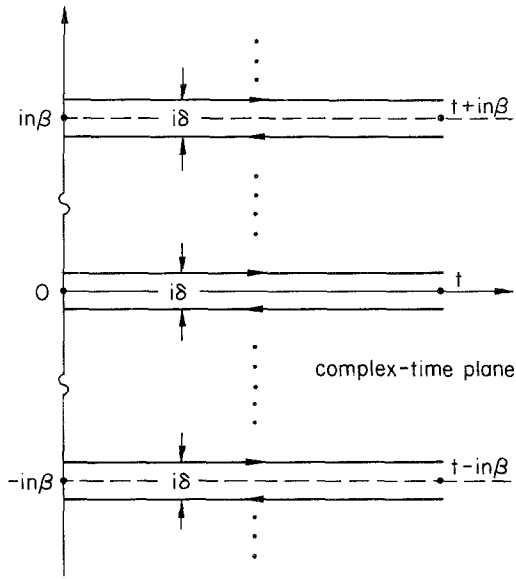

Fig. 3. The contour L'. It is formed by a set of duplications of the contour L along the imaginary axis with a separation $i\beta$.

When z approaches any point of the branch cuts L' [approach to the left-(right-) hand side is marked with the superscript "+ (−)"], we have

$$\Phi^+(\tau) - \Phi^-(\tau) = B(\tau)\,\psi(\tau, \tau') \tag{B5a}$$

$$\Phi^+(\tau) + \Phi^-(\tau) = \psi(\tau, \tau') - \psi^0(\tau, \tau') \tag{B5b}$$

In trying to solve Eqs. (B4) and (B5), we introduce another sectional holomorphic function

$$\begin{aligned} X(z) &\equiv \exp\left(-\frac{1}{2\pi i}\int_{L'} \frac{d\tau''}{\tau'' - z} \ln\frac{1 + B(\tau'')}{1 - B(\tau'')}\right) \\ &\equiv \exp\left[\frac{1}{2\pi i}\int_L d\tau'' \frac{\pi}{\beta}\coth\frac{\pi}{\beta}(\tau'' - z)\ln\frac{1 - B(\tau'')}{1 + B(\tau'')}\right] \end{aligned} \tag{B6}$$

which satisfies

$$\frac{X^+(\tau)}{X^-(\tau)} = \frac{1 - B(\tau)}{1 + B(\tau)}, \qquad \tau \in L' \tag{B7a}$$

$$X(z + in\beta) \equiv X(z) \tag{B7b}$$

Combinings Eqs. (B5)–(B7), we see that

$$\Phi^+(\tau)\,X^+(\tau) - \Phi^-(\tau)\,X^-(\tau) = \frac{X^+(\tau)\,B(\tau)}{1 - B(\tau)}\,\psi^0(\tau, \tau') \tag{B8}$$

This gives an obvious method for constructing $\Phi(z)$ [Eq. (108.9) of Ref. 15],

$$\begin{aligned} \Phi(z) &= \frac{[X(z)]^{-1}}{2\pi i}\int_{L'} \frac{d\tau''\,X^+(\tau'')\,B(\tau'')}{(\tau'' - z)[1 - B(\tau'')]}\,\psi^0(\tau'', \tau') \\ &= \frac{[X(z)]^{-1}}{2\pi i}\int_L d\tau'' \frac{X^+(\tau'')\,B(\tau'')}{1 - B(\tau'')}\frac{\pi/\beta}{\sinh(\pi/\beta)(\tau'' - z)}\,\psi^0(\tau'', \tau') \end{aligned} \tag{B9}$$

The function $\Phi(z)$ clearly satisfies $\Phi(z + in\beta) = (-1)^n\,\Phi(z)$; thus, a formal solution of $\psi(\tau, \tau')$ is found,

$$\begin{aligned} \psi(\tau, \tau') = {} & \frac{1}{1 - B^2(\tau)}\,\psi^0(\tau, \tau') + \frac{1}{\pi i X^+(\tau)[1 + B(\tau)]} \\ & \times \int_L d\tau'' \frac{X^+(\tau'')\,B(\tau'')}{1 - B(\tau'')}\,P\frac{\pi/\beta}{\sinh(\pi/\beta)(\tau'' - \tau)}\,\psi^0(\tau'', \tau) \end{aligned} \tag{B10}$$

To be complete, an important property of the function $X(z)$ should be given here, according to the nature of the principal integral,

$$X^+(\tau)+X^-(\tau)-2=\frac{P}{\pi i}\int_{L'} d\tau' \frac{X^+(\tau')-X^-(\tau')}{\tau'-\tau}$$

$$=\frac{P}{\pi i}\int_{L} d\tau'[X^+(\tau')-X^-(\tau')]\frac{\pi}{\beta}\coth\frac{\pi}{\beta}(\tau'-\tau) \qquad \text{(B11a)}$$

$$\frac{1}{X^+(\tau)}+\frac{1}{X^-(\tau)}-2=\frac{P}{\pi i}\int_{L} d\tau'\left[\frac{1}{X^+(\tau')}-\frac{1}{X^-(\tau')}\right]\frac{\pi}{\beta}\coth\frac{\pi}{\beta}(\tau'-\tau) \qquad \text{(B11b)}$$

B.2. Solution of Equations (28)

Having obtained the solution to Eqs. (27), we now turn to Eqs. (28) for the short-time regime. From Eqs. (26′) (27a), and (B10), we notice the following simple properties[13]:

$$D^0(\tau,\tau')=\begin{cases}O(\delta), & |\tau-\tau'|\gg\delta\\ O(\delta^{-1}), & |\tau-\tau'|\leqslant\delta\end{cases} \qquad \text{(B12a)}$$

$$\oint D^0(\tau,\tau')\,dr=\oint(\tau-\tau')\,D^0(\tau,\tau')\,d\tau=O(\delta) \qquad \text{(B12b)}$$

These can be visualized more easily by looking at the plots of $\tilde{G}^0(\tau,\tau')$ in Ref. 9. What is more important is that $D(\tau,\tau')$ also has the same features when we iterate Eq. (28b). Therefore, we are mainly dealing with a problem such that if $O(\delta)$ is neglected, we can omit all the boundary complications and utilize the technique of Fourier transform.

An easy order-of-magnitude analysis based on Eqs. (B12) shows that the first-order and only the first-order variations of all those slowly varying functions near τ' (on both branches) should also be considered carefully. Furthermore, we notice that there are actually two functions to be solved for a given τ', according to whether τ and τ' are on the same branch. For algebraic convenience, we shall assume τ' is on the upper branch, separate $D(\tau,\tau')$ and all the related functions into two branches, and extend the limits of all the integrals involved to infinity. Now employ the following notations with $t\equiv\tau-\tau'$ and for $\tau\in(\notin)$ the upper branch:

$$D^0(\tau,\tau')=D^0_{1(2)}(t), \qquad \tilde{G}^0(\tau,\tau')=G^0_{1(2)}(t)$$

$$D(\tau,\tau')=D_{1(2)}(t), \qquad F(z(\tau))=F_{1(2)}(t)$$

[13] Note that all the Green's functions have the following simple feature: Within $|\tau-\tau'|\leqslant\delta$ they are proportional to δ^{-1}; otherwise, they are of $O(\delta^0)$. Integrals over one end of these functions can give at most $O(\delta^0)$.

Then Eq. (28c) can be rewritten in the form [cf. Eq. (26)]

$$D_1(t) = D_1^0(t) + \int_{-\infty}^{+\infty} dt'\, G^0(t-t')\, F_1(t')\, D_1(t') - \int_{-\infty}^{+\infty} dt'\, [G_2^0(t-t')]^*\, F_2(t')\, D_2(t') \tag{B13a}$$

$$D_2(t) = D_2^0(t) + \int_{-\infty}^{+\infty} dt'\, G_2^0(t-t')\, F_1(t')\, D_1(t') - \int_{-\infty}^{+\infty} dt'\, [G_1^0(t-t')]^*\, F_2(t')\, D_2(t') \tag{B13b}$$

where the asterisk stands for the complex conjugate. As we have explained, only the first-order variations of $F_1(t)$, $F_2(t)$ should be taken onto account; any higher order contributions would give at most $O(\delta)$. The Fourier transforms of Eqs. (B13) can be readily written down:

$$D_1(\omega) = D_1^0(\omega) + G_1^0(\omega)[F_1 D_1(\omega) - i\dot{F}_1 D_1'(\omega)] - [G_2^0(-\omega)]^*\,[F_2 D_2(\omega) - i\dot{F}_2 D_2'(\omega)] \tag{B14a}$$

$$D_2(\omega) = D_2^0(\omega) + G_2^0(\omega)[F_1 D_1(\omega) - i\dot{F}_1 D_1'(\omega)] - [G_1^0(-\omega)]^*\,[F_2 D_2(\omega) - i\dot{F}_2 D_2'(\omega)] \tag{B14b}$$

Since the relevant frequency scale for this short-time problem is $\omega \sim \delta^{-1}$, $D_1'(\omega)$ and $D_2'(\omega) \sim O(\delta)$ are higher order terms compared to $D_1(\omega)$ and $D_2(\omega)$ of $O(\delta^0)$. Keeping the order of magnitude in mind, we can easily get the answer for Eqs. (B13). Before performing the calculation, let us notice another very important feature: From Eq. (26) we see, for $\beta \to \infty$,

$$\begin{aligned} G_2^0(\omega) &= -iC\rho(\omega)[1 - f(\omega)] \to -iC\rho_0\theta(+\omega)\, e^{-\delta|\omega|} \\ [G_2^0(-\omega)]^*\, G_2^0(\omega) &= C^2\rho_0^2 e^{-2\delta|\omega|} f(-\omega)\, f(\omega) \to 0 \end{aligned} \tag{B15}$$

This enables us to decouple $D_1(\omega)$ from $D_2(\omega)$ because $D_2^0(\omega)$, from Eqs. (28c) and (B10), is proportional to $G_2^0(\omega)$. Thus, Eq. (B14a) simply gives[14]

$$D_1(\omega) = \frac{D_1^0(\omega)}{1 - F_1 G_1^0(\omega)} - i\dot{F}_1 \frac{G_1^0(\omega)}{1 - F_1 G_1^0(\omega)} \frac{d}{d\omega} \frac{D_1^0(\omega)}{1 - F_1 G_1^0(\omega)} + O(\delta^2) \tag{B16}$$

This serves as one of the bases to evaluate the equal-time limit $\tilde{G}(\tau, \tau^+)$.

[14] Concerning the decoupling, care should be taken with regard to terms containing $G_2^{0\prime}(\omega) \to \delta(\omega)$. But this $\delta(\omega)$ does not have any effect, since $D_1^0(\omega)|_{\omega=0} = D_2^0(\omega)|_{\omega=0} = 0$, so that the functions multiplied by $G_2^{0\prime}(\omega)$ always vanish at $\omega = 0$. This could be understood from the point of view of the short-time limit.

B.3. Evaluation of Equal-Time Limit Quantities

At this stage, the main approach toward the solution of Eq. (25) has been finished. Our next goal is to evaluate tr log **G** from the solutions (B16) and (B10). The algebra is quite lengthy but straightforward. First, let us separate $\tilde{G}_L(\tau, \tau')$ into two parts according to the long-time and short-time contributions to $\tilde{G}_L(\tau, \tau^+)$. From Eqs. (B10) and (B7a), recalling the identity (this identity is easily visualized because when $\tau = \tau'$ an integral over τ'' diverges; thus, an extra double δ-function is introduced to cancel this effect; see Refs. 7, 14, and 15)

$$\frac{P}{(\tau''-\tau)}\frac{P}{(\tau''-\tau')} = \frac{P}{\tau-\tau'}\left(\frac{P}{\tau''-\tau} - \frac{P}{\tau''-\tau'}\right) + \pi^2\delta(\tau'', \tau')\,\delta(\tau, \tau')$$

we have

$$\begin{aligned}\tilde{G}^s_L(\tau, \tau') = {} & \frac{1}{1-B^2(\tau)}\left[\tilde{G}^0(\tau, \tau') + i\pi B(\tau)\,\delta(\tau, \tau')\right] \\ & + \frac{1}{i\pi X^+(\tau)[1+B(\tau)]} \\ & \times \int_L d\tau'' \frac{X^+(\tau'')\,B(\tau'')}{1-B(\tau'')} P\frac{\pi/\beta}{\sinh(\pi/\beta)(\tau''-\tau)} \\ & \times \left[\tilde{G}^0(\tau'', \tau') - G^0_L(\tau'', \tau')\right] \qquad \text{(B17a)}\end{aligned}$$

$$\begin{aligned}\tilde{G}^I_L(\tau, \tau') = {} & -\frac{C\rho_0}{4\pi i B(\tau)}\left[\frac{1}{X^+(\tau)} - \frac{1}{X^-(\tau)}\right] \\ & \times \int_L \frac{d\tau''}{2}\left[X^-(\tau'') - X^+(\tau'')\right] \\ & \times \frac{\pi/\beta}{\sinh(\pi/\beta)[(\tau-\tau')/2]}\frac{1}{\cosh(\pi/\beta)[\tau''-(\tau+\tau')/2]} \\ & \times \left(P\frac{\pi/\beta}{\sinh(\pi/\beta)(\tau''-\tau)} - P\frac{\pi/\beta}{\sinh(\pi/\beta)(\tau''-\tau')}\right) \qquad \text{(B17b)}\end{aligned}$$

Obviously $\tilde{G}^s_L(\tau, \tau^+)$ involves only quantities near τ, because the integral over τ'' is cut at long time. The advantage of this separation lies in the fact that the contribution to tr log **G** from the second part can be exactly evaluated via Eq. (B11a), and the contribution from the first part will be

considered by using the method of Fourier transform together with the rest of $\tilde{G}(\tau, \tau')$. Consider the quantity

$$\begin{aligned}
&\int_L d\tau\, F(z(\tau))\, \tilde{G}_L^I(\tau, \tau^+) \\
&\quad = -\int_L \frac{1}{4(\pi i)^2}\left[\frac{1}{X^+(\tau)} - \frac{1}{X^-(\tau)}\right] \\
&\qquad \times \frac{d}{d\tau}\int_L d\tau'\, P\left(\frac{\pi}{\beta}\right) \coth\frac{\pi}{\beta}(\tau'-\tau)[X^-(\tau') - X^+(\tau')] \\
&\quad = \int_L \frac{d\tau}{4\pi i}\left[\frac{1}{X^+(\tau)} - \frac{1}{X^-(\tau)}\right]\frac{d}{d\tau}[X^+(\tau) + X^-(\tau)] \qquad \text{(B18)}
\end{aligned}$$

Recalling Eq. (B6) for $X^{+(-)}(\tau)$ and Eqs. (12), (13), and (23) for the definition of $\tilde{G}_g$, after some algebra, we get the contribution to tr log **G** from $\tilde{G}_I^0(\tau, \tau')$:

$$\begin{aligned}
&\int_0^1 dg \int_L d\tau\, F(z(\tau))\, [\tilde{G}_L^I(\tau, \tau^+)]_g \\
&\quad \equiv \int_0^C \frac{dC}{C}\int_L d\tau\, F(z(\tau))\, \tilde{G}_L^I(\tau, \tau^+) \\
&\quad = \frac{1}{\pi}\int_L d\tau \frac{d}{d\tau}\left[-\int_0^{Z(\tau)} \frac{z\, dz}{\sin 2z} + \frac{Z(\tau)}{2}\right] \\
&\qquad - \frac{1}{2\pi^2}\int_L d\tau\, d\tau'\, Z(\tau)\, Z(\tau')\frac{d}{d\tau}\left[P\left(\frac{\pi}{\beta}\right)\coth\frac{\pi}{\beta}(\tau'-\tau)\right] \qquad \text{(B19a)}
\end{aligned}$$

where we have defined

$$Z(\tau) \equiv \arctan[\pi\rho_0 C z(\tau)] \qquad \text{(B19b)}$$

To deal with the rest of the parts, we shall need the Fourier transform of $\tilde{G}_L^s(\tau, \tau')$ near τ', which is, from Eq. (B17a) for $\tau, \tau' \in$ the same branches [from now on, we neglect the subscript "1" in Eq. (B16), since we no longer restrict τ' to the upper branch]

$$G_L^s(\omega) = G^0(\omega)\frac{1 + FG_L^0(\omega)}{1 + (\pi\rho_0 CF)^2} + \frac{d}{d\omega}\left[-iG^0(\omega)\frac{d}{d\tau'}\frac{1 + FG_L^0(\omega)}{1 + (\pi\rho_0 CF)^2}\right] + O(\delta^2) \qquad \text{(B20)}$$

where

$$F \equiv F(z(\tau')), \qquad G_L^0(\omega) = -i\pi\rho_0 C\, \mathrm{sgn}\,\omega + O(\beta^{-1})$$

Note that during the calculations leading to Eq. (B20), some $G_L^{0\prime}(\omega)$ has

been ignored because of the factor $[G^0(\omega)-G_L^0(\omega)]$. From Eqs. (B20) and (28c), one can also get

$$D^0(\omega)=[G^0(\omega)-G_L^0(\omega)]\left\{G^0(\omega)\frac{F+F^2G_L^0(\omega)}{1+(\pi\rho_0CF)^2}\right.$$
$$\left.+\frac{d}{d\omega}\left[-iG^0(\omega)\frac{d}{d\tau'}\frac{F+F^2G_L^0(\omega)}{1+(\pi\rho_0CF)^2}\right]\right\}+O(\delta^2) \quad \text{(B21)}$$

Then, it is not difficult to find, from Eq. (B16),

$$G_L^s(\omega)+D(\omega)=\frac{G^0(\omega)}{1-FG^0(\omega)}$$
$$-i\frac{\dot F}{2}\left\{\frac{d}{d\omega}\left[\frac{G^0(\omega)}{1-FG^0(\omega)}\right]^2\right.$$
$$\left.+\frac{G^0(\omega)}{1-FG^0(\omega)}\frac{d}{d\omega}\frac{G_L^0(\omega)}{1-FG_L^0(\omega)}\right\}+O(\delta^2) \quad \text{(B22)}$$

The first term is what would have been found had we simply used the adiabatic approximation[7,9]; the rest of the terms are the first-order corrections to the adiabatic result. The equal-time limit is given by

$$\tilde G_L^s(\tau,\tau^+)+D(\tau,\tau^+)$$
$$=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{i0^+}\frac{1}{[G^0(\omega)]^{-1}-F(z(\tau))}$$
$$+\frac{(\pi\rho_0C)^3}{\pi}\frac{\dot FF}{[1+(\pi\rho_0CF)^2]^2} \quad \text{(B23)}$$

where we have used the fact that $G^0(\omega)|_{\omega\to 0}\to -i\pi\rho_0C\,\mathrm{sgn}\,\omega$. Again, the first integral is the adiabatic result,[9] which depends on the form of the large-frequency cutoff of the environmental spectrum, and contributes an adiabatic potential to the bare action of the particle. The second term contributes boundary terms to tr log **G**, which cancel the boundary terms appearing in Eq. (B19a) *exactly* (this cancellation is by no means obvious at the beginning of the computation).

B.4. About the Choices for the Cutoff Procedures

Next, let us return to the general situation where a different cutoff form than $e^{-\delta|\varepsilon|}$ may play a role. A better way to look into the possible effect of this change is to redefine Eq. (27a) for $\tau,\tau'\notin$ same branches,

$$\tilde G_L^0(\tau,\tau')\equiv -C\rho_0\frac{\pi/\beta}{\sinh(\pi/\beta)[\tau-\tau'-i\delta\,\mathrm{sgn}_\gamma(\tau,\tau')]} \quad \text{(B24)}$$

where δ now is a time scale of the order of ω_c^{-1} ($\omega_c \to$ cutoff frequency). As a consequence, $\tilde{G}^0(\tau, \tau') - \tilde{G}_L^0(\tau, \tau') = \Delta G(\tau, \tau') \not\equiv 0$ for $\tau, \tau' \notin$ same branches. This ΔG is, however, very much similar to $G_2^0(t)$ in Eq. (B13), except that the Fourier transform $\Delta G(\omega) \to 0$ at $\omega \to 0$ smoothly. The main property of ΔG is that $\Delta G(\omega)$ is only nonvanishing when $\omega > 0$ or $\omega < 0$, so that any relevant integrals associated with it can be easily shown to be zero by taking the Fourier transform. For these reasons, one can verify that it does not affect the decoupling of $D_1(\omega)$ from $D_2(\omega)$, nor cause $D_1^0(\omega)$ and $\tilde{G}_L^s(\omega)$ to vary, and therefore it does not change the final results (B19) and (B23). Namely, our result is generally valid.

Finally, to get a qualitative feeling for the adiabatic potential due to Eq. (B23), let us choose another density of states for merely analytic convenience, $\rho(\varepsilon) = \rho_0 \omega_c^2/(\varepsilon^2 + \omega_c^2)$. Then

$$G^0(\omega) = \frac{\pi \rho_0 C \omega_c}{\omega + i\omega_c \operatorname{sgn} \omega}$$

This gives the nice adiabatic potential (29), where the trick (13) has been employed (cf. Refs. 7 and 9 for details)

APPENDIX C. BRIEF DERIVATION OF THE EFFECTIVE EUCLIDEAN ACTION (43)

By our path integral approach, we shall derive the effective Euclidean action for the particle alone in parallel to our work on the real-time dynamics. Start with the partition function of the system-plus-environment given by

$$Z(\beta) = \operatorname{Tr}\{e^{-\beta \tilde{H}}\} \tag{C1}$$

In almost exact analogy to the derivation in Appendix A, this partition function is straightforwardly cast into the following functional integral:

$$Z(\beta) = \int \mathscr{D}z(\tau) \prod_{i=1}^{\Omega} \mathscr{D}b_i^\dagger(\tau)\, \mathscr{D}b_i(\tau) \exp\left(-\int_0^\beta d\tau \left[\frac{1}{2} m\dot{z}^2 + V(z) \right.\right.$$
$$\left.\left. + \sum_{i=1}^{\Omega} b_i^\dagger (i\partial_\tau + \varepsilon_i)\, b_i + \sum_{i,j=1}^{\Omega} C_{ij}(z)\, b_i^\dagger b_j + H_{\text{int}}(\{b_i^\dagger, b_i\}) \right] \right) \tag{C2}$$

with the boundary conditions

$$z(0) = z(\beta), \qquad b_i(0) = -b_i(\beta), \qquad b_i^\dagger(0) = -b_i(\beta)$$

For the single-channel problem mentioned in Section 3, when the functional integral over the fermion variables is performed, Eq. (C2) becomes

$$Z(\beta)=\int_{z(0)=z(\beta)} \mathscr{D}z \exp\left\{-\int_0^\beta d\tau \left[\frac{1}{2}m\dot{z}^2+V(z)\right]-\operatorname{tr}\log \mathbf{g}\right\} \tag{C3}$$

where $\mathbf{g}$ is the matrix notation of the temperature Green's function satisfying

$$\mathbf{g}=\mathbf{g}^0-\mathbf{g}^0\mathbf{C}\mathbf{g} \tag{C4}$$

In Eq. (C4), $\mathbf{C}=CF(z(\tau))\,\delta(\tau-\tau')$ and $\mathbf{g}^0$ is given by

$$g^0_{ij}(\tau,\tau')\equiv\delta_{ij}\{\exp[-\varepsilon_i(\tau-\tau')]\}[\theta(\tau-\tau')-f(\varepsilon_i)] \tag{C5}$$

Again, what we are interested in is the tracelike quantity

$$\tilde{g}\equiv C\sum_{i=1}^{\Omega} g_{ii} \tag{C6a}$$

with

$$\tilde{g}^0=C\rho_0\frac{\pi/\beta}{\sin[(\pi/\beta)\,\delta\,\operatorname{sgn}(\tau-\tau')+\tau-\tau']} \tag{C6b}$$

for the choice $\rho(\varepsilon)=\rho_0 e^{-\delta|\varepsilon|}$. This makes Eq. (C4) into the particular type of integral equation encountered above. The detailed algorithm goes exactly parallel to those in Sections 2 and 4 and Appendix B except that the contour and the form of the functions here are much easier to deal with. In fact, the process given by Hamann[7] is more easily understood, though slightly unrigorous. Both of these arrive at the expression[15]

$$Z(\beta)=\int_{z(0)=z(\beta)} \mathscr{D}z \exp\{-S_{\text{eff}}[z(\tau)]\} \tag{C7}$$

where the effective action obtained is given by Eq. (40) in the text.

ACKNOWLEDGMENTS

The author especially acknowledges Prof. A. J. Leggett for his stimulating inspiration, helpful suggestions, and his critical reading of the manuscript, and Dr. C. Yu for thoughtful discussions. He also thanks P. Ao

[15] The expression obtained by Hamann at first sight is not quite the form given here. But a straightforward algebra leads from his result to ours.

for drawing his attention to the use of the path integral and Grassmann algebra. This work was supported by the National Science Foundation through grant DMR 83-15550.

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